"Interstellar" and Black Holes

Makers of the movie "Interstellar" boasted how much they tried to be faithful to physics of gravity. One interesting part of the plot is centered on the Miller planet entirely covered by ocean, orbiting just outside of a giant black hole called "Gargantua." Those who visited its surface, in a small landing ship, finds upon their return to the mother ship that 23 earth-years has passed during their few hour of absence due to the time-dilating effect of gravity. Some journalists criticized the movie for not making much sense, referring to "extremely strong gravitational force, necessary for the extreme time dilation." To see the potential fallacies committed by the film and also by these journalists, and also to understand other important physics aspects of the movie, you may consider the following:

1) Motion of an object of mass m around a non-rotating black hole of mass M (Gargantua is supposed to be a fast-rotating black hole but, here, for simplicity we will assume otherwise) follows Lagrange equation of motion with the action

$$-mc^2 \int ds \sqrt{\left(1-\frac{2GM}{c^2 r}\right)\dot{t}^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2/c^2 - r^2\left(\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2\right)/c^2}$$

c is the speed of light and G is the Newton's gravitational constant. s is an arbitrary parametrization of the motion, $t(s), r(s), \ldots$, with respect to which the "time" derivative as in $\dot{t}, \dot{r}, \ldots$ is taken. You may wish to confine your attention to $r > r_H \equiv 2GM/c^2$, below which things become really weird. The name "Gargantua" refers to the movie fact that the black hole is so heavy that r_H is extremely large, about a billion kilometers.

Equivalently, one may choose to fix the "time" parameter s to be of special type, upon which one is allowed to solve for Lagrange equation of

$$-\frac{1}{2}\int ds \left[\left(1-\frac{r_H}{r}\right)\dot{t}^2 - \left(1-\frac{r_H}{r}\right)^{-1}\dot{r}^2/c^2 - r^2\left(\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2\right)/c^2\right]$$

instead. This should be a little easier to handle. Show why solving this action principle is equivalent to solving the first action principle. Describe what is the residual freedom on s, which used to be an arbitrary parameter in the first action principle.

2) List all symmetries and conserved quantities and solve for the orbit. For generic orbits, there is no analytic solutions, so qualitative characterization or numerical

plotting will do. This way, you end up with a first order differential equation, again as in the Newtonian Kepler problem. With these, classify possible orbits of the planet, which should be similar to Kepler ones when $r \gg r_H$ but qualitatively different when $r/r_H < 10$ or so. What is the smallest possible circular orbit, for example? Where did you use the mass of the test particle? Repeat the exercise for massless particles.

3) The time lapse felt by different observers are different, depending on position and on velocity. When a far away observer feels time lapse of δt, the person moving near black hole feels, instead, a smaller time lapse of

$$\delta\tau = \delta t \times \sqrt{\left(1 - \frac{r_H}{r}\right) - \left(1 - \frac{r_H}{r}\right)^{-1} \dot{r}^2/(c^2\dot{t}^2) - r^2\left(\dot{\theta}^2 + (\sin\theta)^2\dot{\phi}^2\right)/(c^2\dot{t}^2)}$$

Now, you are in position to address physics of the Miller planet and the plot around it. What is the maximum possible time-dilation can you imagine for a reasonable, in the context of the movie, orbit of the planet around Gargantua? Does a large time dilation really mean a very "strong gravitational force"? Why did the producers choose a very large black hole, do you think? What important feature of the Gargantua are we forgetting about here?

4) The black hole is black, we often say, because nothing can escape from it, or more precisely nothing inside $r < r_H$ reach us at $r > r_H$, not even lights, while objects at $r > r_H$ can easily fall inside. However, the equation of motion you would find from the action principle in 1) has two of the s-derivative in all terms, so is invariant under $s \to -s$. This means that if there is an infalling trajectory, there must be its s-reversed trajectory that describes something at $r < r_H$ moving out to $r > r_H$. This seemingly contradicts the statement that nothing can escape from $r < r_H$. Please investigate why there is actually no contradiction, or why the black hole is really black. One should be cautioned that this apparent quandary cannot be resolved properly within this relativistic Kepler problem.

5) Another a key plot device is a "traversable wormhole" that may allow one to reach a very far away place at an effective speed far greater than the speed of light. Research existing physics literatures on wormholes and discuss known limitations.

This project has a lot of components. Instead of following the items linearly as if it is a homework, you are free to skip and choose, especially if you already understand the easier part from previous study, as long as the final result to be presented is substantial.