The Physics of Neutrino Oscillation

History of Neutrino Oscillations

- The first idea of NO was considered by B. Pontecorvo
  $\nu \leftrightarrow \bar{\nu}$ (1957)
- Mixing was introduced at the beginning of 60's
  (Maki, Nakagawa & Sakata (1962), cf) Cabibbo (1963))
- The first computation of probability was performed
  (Gribov & Pontecorvo (1969))
- The first indication of NO came from solar neutrino
  (Davis, Homestake, found solar ν flux deficit (1964,68))
- An anomaly was found in atmospheric neutrinos
  (Superkamiokande (1998)

Neutrino Oscillation

- Neutrino quantum states produced via weak interaction

  → flavor eigenstates

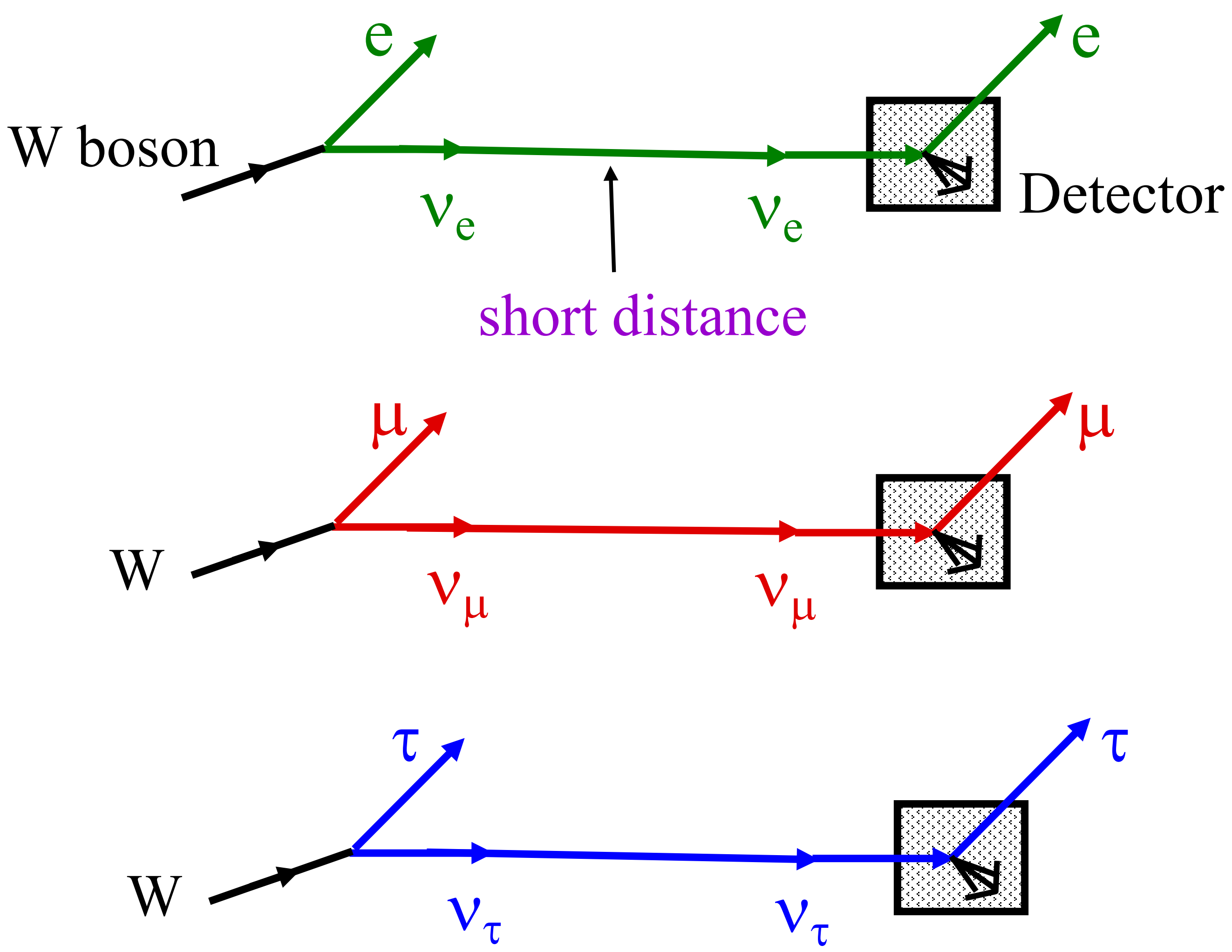
- There are 3 flavor eigenstates, and each of these is associated with the corresponding lepton- flavor.

$$\nu_e \;,\; \nu_\mu \;,\; \nu_\tau$$

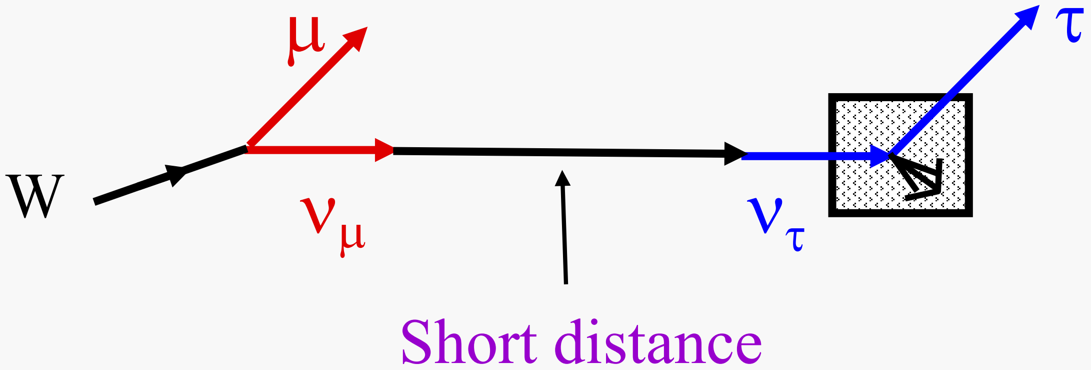
$$\updownarrow \qquad \updownarrow \qquad \updownarrow$$

$$e \;,\; \mu \;,\; \tau$$

- **Neutrino flavor와 charged-lepton flavor 사이의 연관성이 의미하는 것 :**



Over short distance, neutrinos do not change flavor



Does not occur

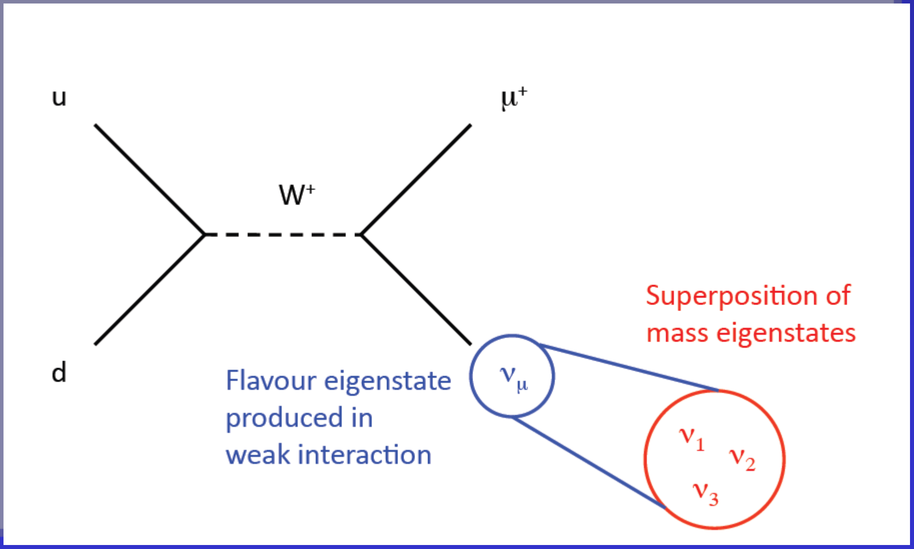
Neutrino Oscillation

- **Flavor states do not coincide with mass states**
- **Flavor states :** $\{ |\nu_e\rangle \quad , \quad |\nu_\mu\rangle \quad , \quad |\nu_\tau\rangle \}$
- **Mass states :** $\{ |\nu_1\rangle \quad , \quad |\nu_2\rangle \quad , \quad |\nu_3\rangle \}$
- Neutrino quantum state produced by weak interactions

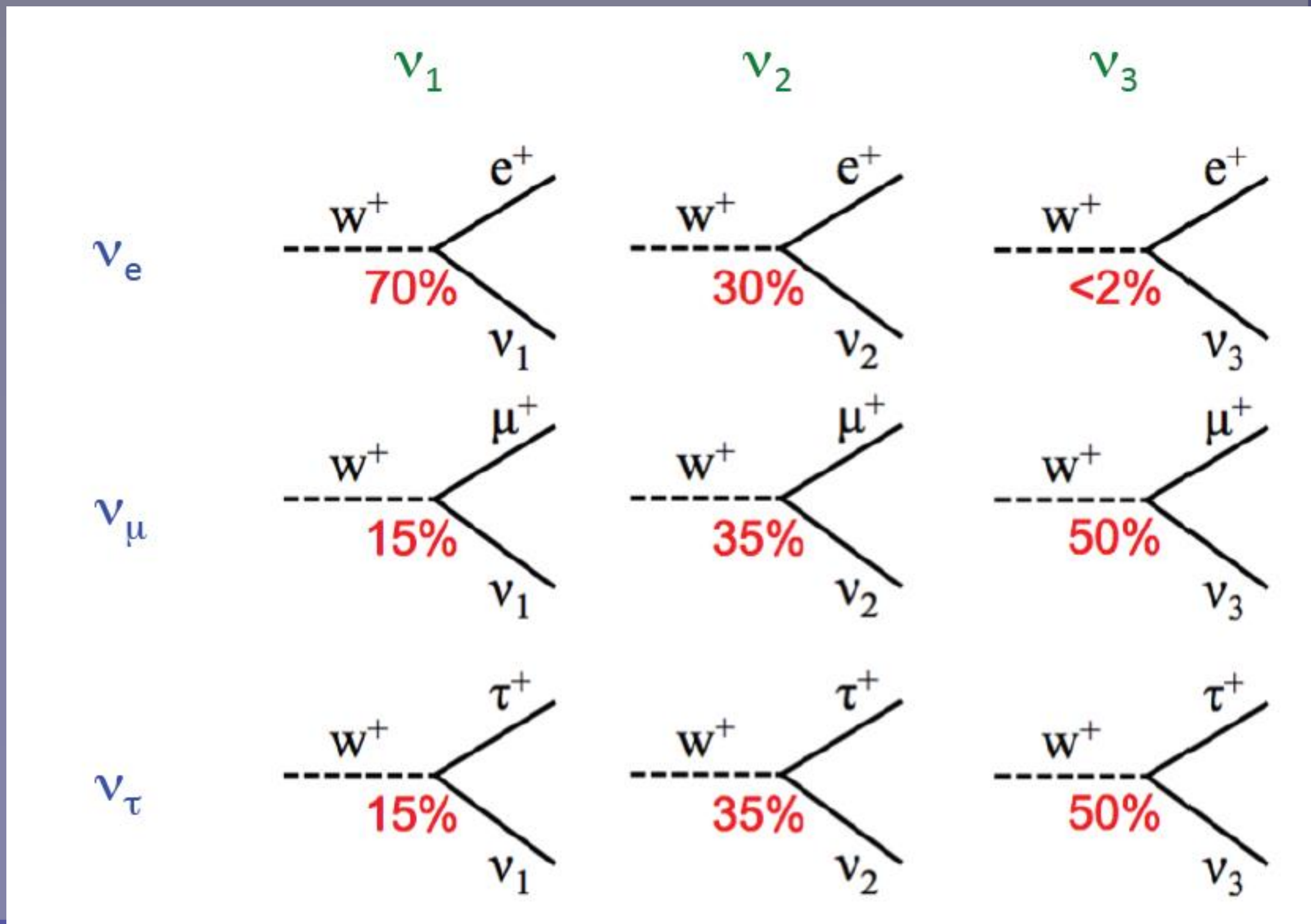
$$|\nu_\alpha> = \Sigma\, U^*_{\alpha i}\, |\nu_i>$$ → mass eigenstates .

flavor α : e, μ, or τ

Mixing matrix



Neutrino Oscillation



Neutrino Oscillation

Fermion Mixing

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = V^{\mathrm{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = U^{\mathrm{PMNS}} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Neutrino Oscillation in Vaccum

- **Quantum mechanical effects when**

Evolution of ν_α :

$$|\nu_k(t,x)\rangle = e^{-iE_k t + ip_k x}\,|\nu_k\rangle \quad \Rightarrow$$

$$|\nu_\alpha(t,x)\rangle = \sum_k U^*_{\alpha k}\, e^{-iE_k t + ip_k x}\,|\nu_k\rangle$$

Neutrino Oscillation

- Non-relativisitic Schrodinger Eq. for a free particle

$$E = \frac{p^2}{2m}.$$

Substituting
$$E \to i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} = -i\hbar \boldsymbol{\nabla}$$

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2m}\nabla^2 \psi = 0$$

- Relativistic quantum equation from energy-momentum rel.

$$E^2 = \mathbf{p}^2 + m^2.$$

$$-\frac{\partial^2 \phi}{\partial t^2} + \nabla^2 \phi = m^2 \phi$$
→ Klein-Gordon Eq.

$$\phi = N e^{i\mathbf{p}\cdot\mathbf{x} - iEt}$$
→free particle solution

Neutrino Oscillation

- **Natural Units : choosing units** $\hbar = c = 1$

$$\hbar = \frac{h}{2\pi} = 1.055 \times 10^{-34} \text{ J sec}$$
$$c = 2.998 \times 10^{8} \text{ m sec}^{-1}$$

$\hbar$ (ML^2/T) and c (L/T)

- **In high E physics, quantities are measured in units of GeV**

  e.g. $m_p \approx 1$ **GeV**

$$m \text{ kg} \equiv mc^2 \text{ Energy units}$$

$$\begin{aligned} 1 \text{ kg} &\equiv 1 \times (2.998 \times 10^8)^2 \text{ J} \\ &= \frac{(2.998 \times 10^8)^2 \text{ J}}{1.6 \times 10^{-19} \frac{\text{J}}{\text{eV}}} \\ &= 5.618 \times 10^{-35} \text{ eV} \\ &= 5.618 \times 10^{-26} \text{ GeV.} \end{aligned}$$

Table 2.3: Conversion factors for MKS to Natural units

Quantity	Conversion factor	Actual dimension
Mass	$1 \text{ kg} = 5.62 \times 10^{-26} \text{ GeV}$	$\frac{\text{GeV}}{c^2}$
Length	$1 \text{ m} = 5.07 \times 10^{15} \text{ GeV}^{-1}$	$\frac{\hbar c}{\text{GeV}}$
Time	$1 \text{ sec} = 1.52 \times 10^{24} \text{ GeV}^{-1}$	$\frac{\hbar}{\text{GeV}}$

Neutrino Oscillation

- **Relativisitic Schrodinger Eq. in a form linear in $\partial/\partial t$**

$$H\psi = (\boldsymbol{\alpha}\cdot\mathbf{P} + \beta m)\psi = i\frac{\partial}{\partial t}\psi$$

- **α_i , β are constants and satisfy** $E^2 = \mathbf{p}^2 + m^2.$

$$H^2\psi = (\mathbf{P}^2 + m^2)\psi$$

- $\alpha_i^2 = \beta^2 = 0$
- $\alpha_i\alpha_j + \alpha_j\alpha_i = \alpha_i\beta + \beta\alpha_i = 0$. Hence α_i's and β anticommute with one another.

- **Due to the last relations, α_i, β cannot be numbers but matirces**

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Neutrino Oscillation

- **Relativisitic Schrodinger Eq. in a form linear in $\partial/\partial t$**

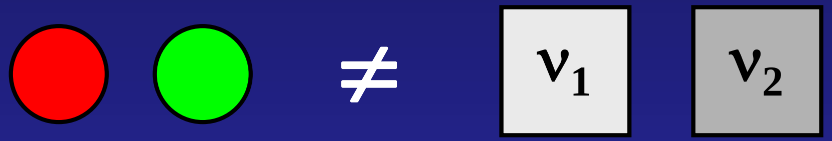
$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

where $\gamma^{\mu} \equiv (\beta, \beta\boldsymbol{\alpha})$ and $\partial_{\mu} = \left(\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right)$ (in four vector notation). γ^{μ}'s are known as the Dirac γ matrices

Neutrino Oscillation in Vaccum

- **Quantum mechanical effects when**

Flavor states Mass states



Evolution of ν_k (mass states) :

$$|\nu_k(t,x)\rangle = e^{-iE_k t + i p_k x}\,|\nu_k\rangle \quad \Rightarrow$$

Evolution of ν_α (flavor states) :

$$|\nu_\alpha(t,x)\rangle = \sum_k U^*_{\alpha k}\, e^{-iE_k t + i p_k x}\,|\nu_k\rangle$$

Neutrino Oscillation

$$|\nu_k\rangle = \sum U_{\beta k}|\nu_\beta\rangle \quad \Rightarrow$$

Plugging into above

$$|\nu_\alpha(t,x)\rangle = \sum_{\beta=e,\mu,\tau} \underbrace{\left(\sum_k U^*_{\alpha k} e^{-iE_k t + i p_k x} U_{\beta k}\right)}_{\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t,x)} |\nu_\beta\rangle$$

$$\mathcal{A}_{\nu_\alpha \to \nu_\beta}(0,0) = \sum_k U^*_{\alpha k} U_{\beta k} = \delta_{\alpha\beta}$$

$$\mathcal{A}_{\nu_\alpha \to \nu_\beta}(t > 0, x > 0) \neq \delta_{\alpha\beta}$$

Neutrino Oscillation

$$P_{\nu_\alpha \to \nu_\beta}(t, x) = \left| \mathcal{A}_{\nu_\alpha \to \nu_\beta}(t, x) \right|^2$$

$$= \left| \sum_k U_{\alpha k}^* e^{-iE_k t + i p_k x} U_{\beta k} \right|^2$$

$$E_k t - p_k x \simeq (E_k - p_k) L = \frac{E_k^2 - p_k^2}{E_k + p_k} L = \frac{m_k^2}{E_k + p_k} L \simeq \frac{m_k^2}{2E} L$$

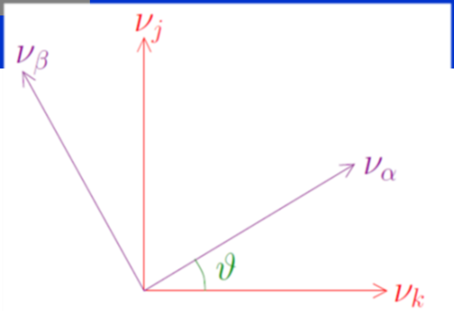
Neutrino Oscillation

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \left| \sum_k U_{\alpha k}^* \, e^{-i m_k^2 L/2E} \, U_{\beta k} \right|^2$$

$$= \sum_{k,j} U_{\alpha k}^* U_{\beta k} U_{\alpha j} U_{\beta j}^* \exp\left(-i \frac{\Delta m_{kj}^2 L}{2E} \right)$$

$$\Delta m_{kj}^2 \equiv m_k^2 - m_j^2$$

2-Neutrino Oscillation Probability



$$P(\nu_\mu \to \nu_\tau; t) =$$

$$\begin{aligned}
&= |\langle \nu_\tau | \nu(t) \rangle|^2 \\
&= |\{-\sin\theta \langle \nu_1| + \cos\theta \langle \nu_2|\}|\{\cos\theta\, e^{-iE_1 t}|\nu_1\rangle + \sin\theta\, e^{-iE_2 t}|\nu_2\rangle\}|^2 \\
&= \cos^2\theta \sin^2\theta \left| e^{-iE_2 t} - e^{-iE_1 t} \right|^2 \\
&= 2\cos^2\theta \sin^2\theta \{1 - \cos[(E_2 - E_1)t\}
\end{aligned}$$

$$= \sin^2 2\theta \sin^2\left[\frac{\Delta m^2}{4E} t\right]$$

In natural unit, t=L

Converting natural unit to lab. unit :

$$\frac{\Delta m^2 L}{4E} = 1.27 \frac{\Delta m^2[\text{eV}^2]\, L[\text{m}]}{E[\text{MeV}]} = 1.27 \frac{\Delta m^2[\text{eV}^2]\, L[\text{km}]}{E[\text{GeV}]}$$

θ

2-Neutrino Oscillation Probability

- ν_e is created at L=0 (t=0) and travels a distance of L

$$P_{e\mu} = |\langle \nu_\mu | \nu_e, t \rangle|^2$$
$$= \sin^2 2\theta \sin^2\left[1.27 \frac{\Delta m^2_{21} L}{E_\nu}\right]$$

Neutrino trajectory

Neutrino energy

2 fundamental parameters (θ Δm^2)

Survival (disappearance) probability

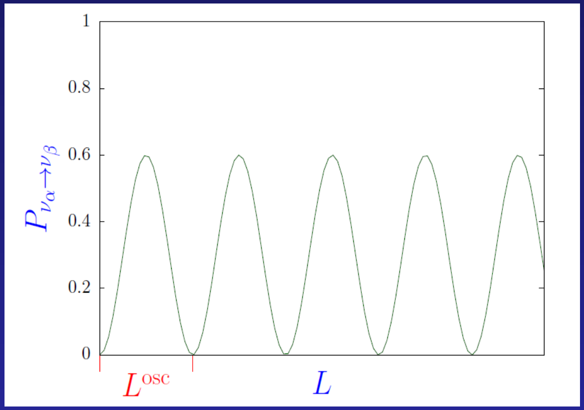
$$P_{ee} = |\langle \nu_e | \nu_e, t \rangle|^2 = 1 - P_{e\mu}$$

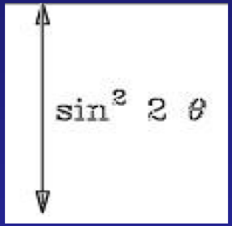
- Oscillations require neutrinos to have different masses ($\Delta m^2 \neq 0$) and to mix ($U_{\alpha i} U^*_{\beta i} \neq 0$).

θ

2-Neutrino Oscillation Probability

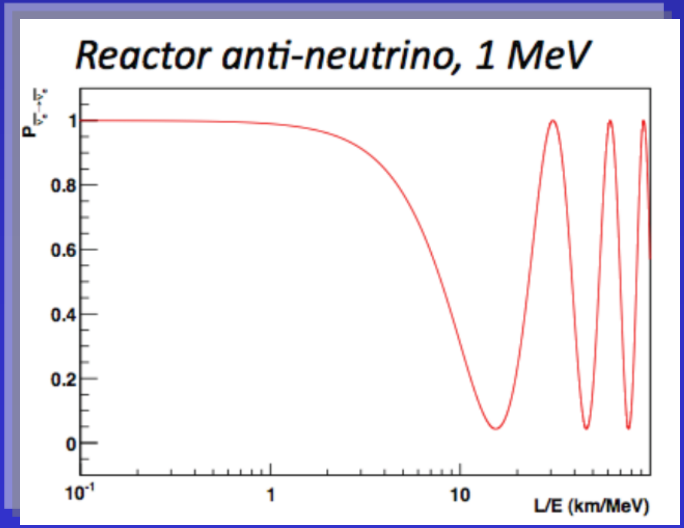
example





- Neutrino survival probability

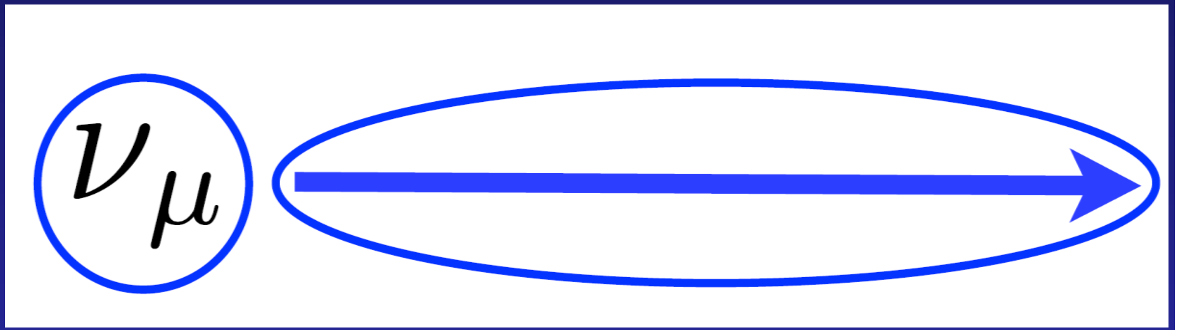
$$P_{survival} = 1 - P_{e\mu}$$

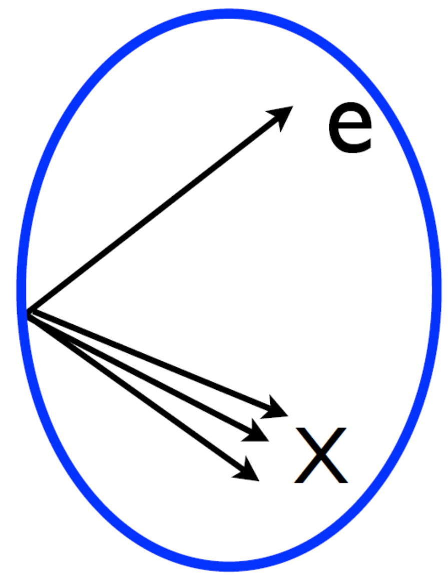


E=1 MeV, θ=40 deg

$\Delta m^2 = 8\times10^{-5}(\text{eV})^2$

Neutrino Oscillation





Production
Flavor state

$$|\nu_\mu\rangle = \sum_i U_{\mu i}|\nu_i\rangle$$

Propagation
Mass state

$$\nu_1 : \quad e^{-iE_1 t}$$
$$\nu_2 : \quad e^{-iE_2 t}$$

Detection
Flavor state

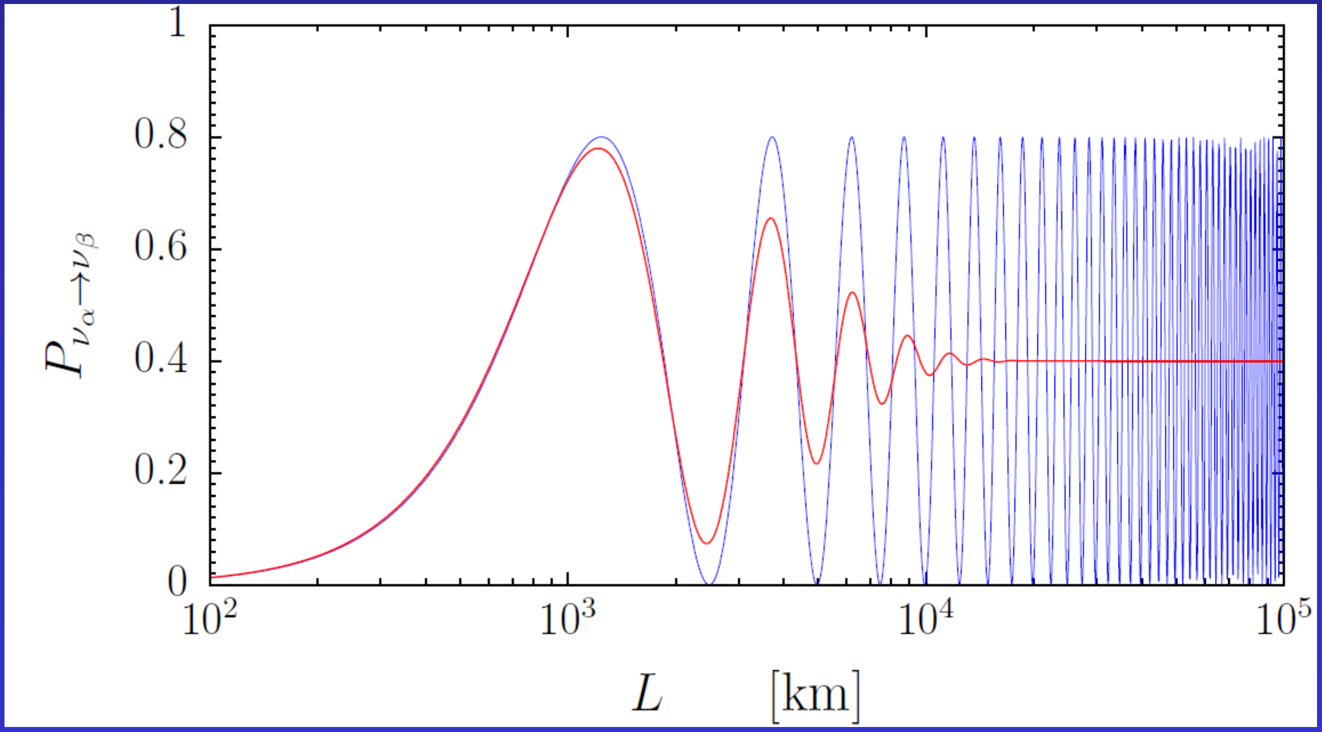
$$\langle \nu_e |$$

θ

Average over energy resolution of detector

$$P_{\nu_\alpha \to \nu_\beta}(L, E) = \sin^2 2\vartheta \sin^2\left(\frac{\Delta m^2 L}{4E}\right) = \frac{1}{2}\sin^2 2\vartheta \left[1 - \cos\left(\frac{\Delta m^2 L}{2E}\right)\right]$$

- $$\langle P_{\alpha\beta} \rangle = \frac{\int dE\nu \, d\Phi/dE\nu \, \sigma_{CC}(E\nu) \, P_{\alpha\beta}(E\nu) \, \varepsilon(E\nu)}{\int dE\nu \, d\Phi/dE\nu \, \sigma_{CC}(E\nu) \, \varepsilon(E\nu)}$$
  $$= \delta_{\alpha\beta} - (2\delta_{\alpha\beta} - 1)\sin^2 2\theta \, \langle \sin^2 (\Delta m^2 L/4E) \rangle.$$

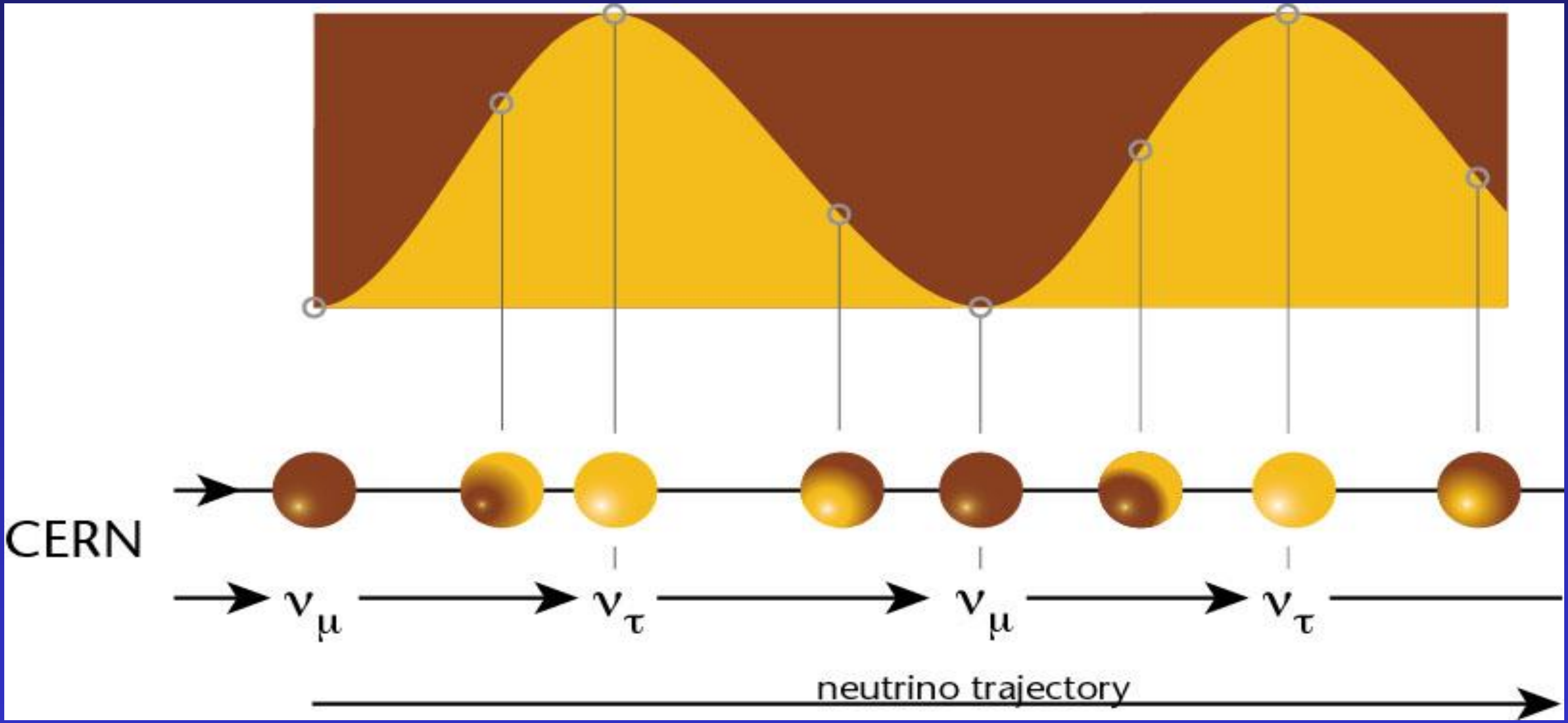


$\Delta m^2 = 10^{-3}\,\text{eV}$

$\sin^2 2\vartheta = 0.8$

$\langle E \rangle = 1\,\text{GeV}$

Two-Neutrino Oscillation



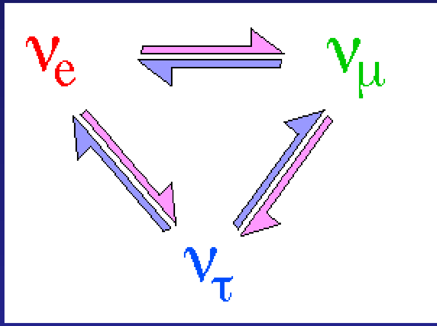
Three Neutrino Oscillations

Neutrino Mixing

$$|\nu_\alpha\rangle = \sum U_{\alpha i} |\nu_i\rangle$$

Weak eigenstate

Mass eigenstate



$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \underbrace{\begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}}_{U_{\text{PMNS}}} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}$$

Pontecorvo-Maki-Nakagawa-Sakata

Neutrino Mixing

- The unitary mixing matrix U occurs in C.C. weak interactions

(in the flavor basis) :

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(\nu_e, \nu_\mu, \nu_\tau)_{\rm L}} \gamma^\mu \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_{\rm L} W_\mu^+ + {\rm h.c.}$$

(in the mass basis) :

$$-\mathcal{L}_{\rm cc} = \frac{g}{\sqrt{2}} \overline{(\nu_1, \nu_2, \nu_3)_{\rm L}} U^\dagger \gamma^\mu \begin{pmatrix} e \\ \mu \\ \tau \end{pmatrix}_{\rm L} W_\mu^+ + {\rm h.c.}$$

Neutrino Mixing matrix

$$U = V_L^{\ell\dagger} V_L^{\nu} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix}$$

- 3×3 unitary mixing matrix depends on 3^2 independent parameters :
  - 3 angles & 6 phases
- Not all phases are physical observables
- Let's see how many phases are physical by assuming neutrinos are Dirac fermion.

Neutrino Mixing matrix

- From charged current : $\sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}}\, U_{\alpha k}^{*}\, \gamma^{\rho}\, \ell_{\alpha L}$

- under global phase transformation:

$$\nu_k \to e^{i\varphi_k}\, \nu_k \quad (k = 1, 2, 3)\,, \qquad \ell_\alpha \to e^{i\varphi_\alpha}\, \ell_\alpha \quad (\alpha = e, \mu, \tau)$$

$$\sum_{k=1}^{3} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}}\, e^{-i\varphi_k}\, U_{\alpha k}^{*}\, e^{i\varphi_\alpha}\, \gamma^{\rho}\, \ell_{\alpha L}$$

$$\underbrace{e^{-i(\varphi_1-\varphi_e)}}_{1} \sum_{k=1} \sum_{\alpha=e,\mu,\tau} \overline{\nu_{kL}}\, \underbrace{e^{-i(\varphi_k-\varphi_1)}}_{2}\, U_{\alpha k}^{*}\, \underbrace{e^{i(\varphi_\alpha-\varphi_e)}}_{2}\, \gamma^{\rho}\, \ell_{\alpha L}$$

- 5 phases can be eliminated by redefining fields

Neutrino Mixing matrix

- The mixing matrix contains 1 physical phase.
- 3x3 unitary mixing matrix can be expressed in terms of 3 mixing angles and 1 phase.

- Standard parameterization

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta_{13}} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta_{13}} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23}-c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23}-s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23}-c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23}-s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}$$