ON THE $\mathcal{Q}^{\perp}$-PARALLELISM OF RICCI TENSORS IN REAL HYPERSURFACES EMBEDDED IN COMPLEX GRASSMANNIANS OF RANK TWO

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Abstract. In this paper, the notions of $\mathcal{Q}^{\perp}$-parallel Ricci tensor is introduced on a real hypersurface M in complex Grassmannians of rank two. Moreover, by using classification theory, we can give a complete classification for real hypersurfaces M in complex Grassmannians of rank two.

1. Introduction

$G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$ has a compact transitive group SU_{2+m}, however $SU_{2,m}/S(U_2{\cdot}U_m)$ has a noncompact indefinite transitive group $SU_{2,m}$. This distinction gives various remarkable results. Riemannian symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ has a remarkable geometrical structure. It is the unique noncompact, Kähler, irreducible, quaternionic Kähler manifold with negative curvature.

Suppose that M is a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$). Let N be a local unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$). Since $G_2(\mathbb{C}^{m+2})$(or $SU_{2,m}/S(U_2{\cdot}U_m)$) has the Kähler structure J, we may define the *Reeb vector field* $\xi = -JN$ and a one dimensional distribution $[\xi] = \mathcal{C}^{\perp}$ where $\mathcal{C}$ denotes the orthogonal complement in T_xM, $x \in M$, of the Reeb vector field ξ. The Reeb vector field ξ is said to be *Hopf* if $\mathcal{C}$ (or $\mathcal{C}^{\perp}$) is invariant under the shape operator A of M. The one dimensional foliation of M defined by the integral curves of ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* if and only if the Hopf foliation of M is totally geodesic. By the formulas, it can be checked that ξ is Hopf vector field if and only if M is Hopf hypersurface.

From the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$), there naturally exist *almost contact 3-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1,2,3$. Put $\mathcal{Q}^{\perp} = \text{Span}\{\,\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent bundle TM of M. In addition, we denoted by $\mathcal{Q}$ the orthogonal complement of $\mathcal{Q}^{\perp}$ in TM. It is the quaternionic maximal subbundle of TM. Thus the tangent bundle of M is expressed as a direct sum of $\mathcal{Q}$ and $\mathcal{Q}^{\perp}$.

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2. PRELIMINARIES

In this section we derive some basic formulas and the Codazzi equation for a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$) (see [2], [4], [6], [7], [8], [9], and [10]).

Let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$). The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M,g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N.

Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.1}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$) there exists an almost contact metric structure (ϕ,ξ,η,g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X,\xi) \tag{2.2}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$ (or $SU_{2,m}/S(U_2{\cdot}U_m)$), together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$ in Section 1, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}
&\phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0,\\
&\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},\\
&\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,\\
&\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{2.3}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1,2,3$ and (2.1), the relation between these two contact metric structures (ϕ,ξ,η,g) and $(\phi_\nu,\xi_\nu,\eta_\nu,g)$, $\nu = 1,2,3$, can be given by

$$\begin{aligned}
&\phi\phi_\nu X = \phi_\nu\phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu,\\
&\eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi.
\end{aligned} \tag{2.4}$$

3. Main Theorem

3.1. **Lemma.** *Let M be a Hopf hypersurface in complex Grassmannians of rank 2, $m \geq 3$. If M has $\mathcal{Q}^\perp$-parallel Ricci tensor, then ξ belongs to either $\mathcal{Q}$ or $\mathcal{Q}^\perp$.*

Hence, from now on we will prove whether a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with our hypotheses given in main theorems satisfy the Ricci $\mathcal{Q}^\perp$-parallel condition.

3.2. **Lemma.** *Let M be a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, with $\mathcal{Q}^\perp$-parallel Ricci tensor. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then $(S\phi-\phi S)X = 0$ on M.*

3.3. **Theorem.** *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Ricci tensor of M is $\mathcal{Q}^\perp$-parallel if and only if M is an open part of a tube of some radius r around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$.*

3.4. **Theorem.** *Let M be a connected orientable real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$. Then the Ricci tensor of M is $\mathcal{Q}^\perp$-parallel if and only if M is an open part of a tube of some radius r around a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$.*

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