# Hopf algebras arising from dg manifolds

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Workshop on Atiyah classes and related topics

# 1. Introduction

In 1957, Atiyah defined an obstruction class, the Atiyah class, to the existence of holomorphic connections on a holomorphic vector bundle.

It plays important roles, for example, in

i). deformation quantization (Kontsevich 2003),

ii). Rozansky-Witten invariants (Kapranov 1999, Kontsevich 1999),

iii). Chern character and Riemann-Roch theorem (Ramadoss 2008, Markarian 2009).

# 1. Introduction

Atiyah classes form a bridge between complex geometry and Lie theory.

$(X, \mathcal{O}_X)$: complex smooth algebraic variety.

The Atiyah class $\alpha_X$ of $T_X$ defines a Lie bracket on $T_X[-1]$.

### Theorem (Ramadoss 2008)

*The universal enveloping algebra*

$$U((T_X[-1]; \alpha_X)) \simeq (D^{\bullet}_{\mathrm{poly}}(X), d_H),$$

*in* $D^+(\mathcal{O}_X)$*. Here* $(D^{\bullet}_{\mathrm{poly}}(X), d_H) =$ *Hochschild cochain complex of polydifferential operators, which is a Hopf algebra.*

We are inspired by Ramadoss's work, and will show a parallel picture in dg geometry context.

# 2. dg manifold: graded manifold

$\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Grading $:=$ $\mathbb{Z}$-grading.

A finite dimensional graded manifold $\mathcal{M}$ is a pair $(M, \mathcal{O}_{\mathcal{M}})$:

i). $M$: smooth manifold, called the support of $\mathcal{M}$,

$\mathcal{O}_{\mathcal{M}}$: sheaf of graded commutative algebra on $M$.

ii). $\exists$ finite dimensional graded $\mathbb{K}$-vector space $V$, s.t.

$\forall x \in M$, $\exists$ open $U \subset M$ , s.t. $\mathcal{O}_{\mathcal{M}}(U) \simeq C^{\infty}(U, \mathbb{K}) \otimes_{\mathbb{K}} \hat{S}(V^{\vee})$.

$C^{\infty}_{\mathcal{M}} := \mathcal{O}_{\mathcal{M}}(M)$, the graded ring of functions on $\mathcal{M}$.

# 2. dg manifold

A degree $k$ vector field $X$ on $\mathcal{M}$ is a $\mathbb{K}$-linear morphism:

$$X : (C^\infty_{\mathcal{M}})^\bullet \to (C^\infty_{\mathcal{M}})^{\bullet+k},$$
$$X(fg) = X(f)g + (-1)^{k\tilde{f}} fX(g).$$

$\mathscr{X}(\mathcal{M}) :=$ all vector fields on $\mathcal{M}$.

A dg manifold is a pair $(\mathcal{M}, Q)$:
— $\mathcal{M}$: a graded manifold,
— $Q$: degree 1 vector field, s.t.

$$Q^2 = 0 \qquad \text{(homological condition)}.$$

# 2. dg manifold: examples

i). $\mathfrak{g}$ : Lie algebra.

$(\mathfrak{g}[1], d_{CE})$ is a dg manifold, $C^\infty_{\mathfrak{g}[1]} = \wedge^\bullet \mathfrak{g}^\vee$,

$$d_{\mathrm{CE}} : \wedge^\bullet \mathfrak{g}^\vee \to \wedge^{\bullet+1} \mathfrak{g}^\vee.$$

ii). $A$: Lie algebroid, by similar construction:

$(A[1], d^A_{\mathrm{CE}})$ is a dg manifold.

iii). Kapranov dg manifolds, Fedosov dg manifolds, etc...

# 3. dg module

$(\mathcal{M}, Q)$: dg manifold.

A dg module over $(\mathcal{M}, Q)$ is a pair $(\mathfrak{N}, L_Q)$:

i). $\mathfrak{N}$ is a graded $C^\infty_{\mathcal{M}}$-module,

ii). $L_Q : \mathfrak{N} \to \mathfrak{N}$ is a degree $+1$ and $\mathbb{K}$-linear map, s.t.

$$L_Q(f\xi) = Q(f)\xi + (-1)^{\widetilde{f}} f L_Q(\xi)$$

$\forall \xi \in \mathfrak{N}, f \in C^\infty_{\mathcal{M}}$.

# 3. dg module

Morphism $\varphi : (\mathfrak{N}_1, L_Q) \to (\mathfrak{N}_2, L_Q)$:

i). $\varphi : \mathfrak{N}_1 \to \mathfrak{N}_2$ is a morphism of $C^\infty_{\mathcal{M}}$-modules, i.e.,

$$\varphi(f\xi) = f\varphi(\xi),$$

$\forall f \in C^\infty_{\mathcal{M}}$ , $\xi \in \mathfrak{N}_1$.

ii).

$$L_Q \circ \varphi = \varphi \circ L_Q.$$

Denote by **dg−mod** the category of dg modules over $(\mathcal{M}, Q)$.

# 4. dg module: examples

i). $(\mathcal{E}, L_Q)$: a dg vector bundle, $(\Gamma(\mathcal{E}), L_Q)$ is a dg module.

ii). $(\mathscr{X}(\mathcal{M}), L_Q = [Q, \cdot])$ is a dg module.

iii). $D_{\mathcal{M}} := U(T_{\mathcal{M}})$, differential operators on $\mathcal{M}$, i.e., the universal enveloping algebra of the Lie algebroid $T_{\mathcal{M}}$.

$(D_{\mathcal{M}}, L_Q)$ is a dg module, which does not correspond to the space of global sections of any dg vector bundle.

# 5. homotopy category

Quasi-isomorphism of dg modules, is a morphism of dg modules

$$\varphi : (\mathfrak{N}_1, L_Q) \to (\mathfrak{N}_2, L_Q), \text{ s.t.}$$

$$H^\bullet(\varphi) : H^\bullet(\mathfrak{N}_1, L_Q) \simeq H^\bullet(\mathfrak{N}_2, L_Q)$$

is an isomorphism.

The homotopy category $\Pi(\mathbf{dg}-\mathbf{mod}) :=$

Gabriel-Zisman localization of $\mathbf{dg}-\mathbf{mod}$

by the set of quasi-isomorphism of dg modules.

The homology category $\mathrm{H}(\mathbf{dg}-\mathbf{mod}) :=$ the category $\mathbf{dg}-\mathbf{mod}$ modulo cochain homotopies.

Sequence of natural functors:

$$\mathbf{dg}-\mathbf{mod} \to \mathrm{H}(\mathbf{dg}-\mathbf{mod}) \to \Pi(\mathbf{dg}-\mathbf{mod}).$$

# 6. dg complex

A dg complex over $(\mathcal{M}, Q)$ is a triple $(\Upsilon^\bullet, L_Q, \delta)$:

i). $\Upsilon^\bullet = \oplus_p \Upsilon^p$,
$L_Q : \Upsilon^p \to \Upsilon^p$, each $(\Upsilon^p, L_Q)$ is a dg module over $(\mathcal{M}, Q)$.

ii). $\delta : \Upsilon^\bullet \to \Upsilon^{\bullet+1}$ is a $C^\infty_{\mathcal{M}}$-linear operator, i.e.

$$\delta(f\xi) = (-1)^{\widetilde{f}} f\delta(\xi), \ \forall f \in C^\infty_{\mathcal{M}}, \ \xi \in \Upsilon,$$

$$[\delta, L_Q] = \delta \circ L_Q + L_Q \circ \delta = 0.$$

iii). $\delta \circ \delta = 0$.

# 6. dg complex

It is convenient to denote such a dg complex by a diagram of double complex:

$$
\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
\cdots & \xrightarrow{\delta} & \Upsilon^{p,q+1} & \xrightarrow{\delta} & \Upsilon^{p+1,q+1} & \xrightarrow{\delta} & \cdots \\
 & & L_Q \Big\uparrow & & L_Q \Big\uparrow & & \\
\cdots & \xrightarrow{\delta} & \Upsilon^{p,q} & \xrightarrow{\delta} & \Upsilon^{p+1,q} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow & & \uparrow & &
\end{array} .
$$

We call $\delta$ the **horizontal differential** and $L_Q$ the **vertical differential**.

# 6. dg complex

Morphism $\varphi : (\Upsilon_1^\bullet, L_Q, \delta_1) \to (\Upsilon_2^\bullet, L_Q, \delta_2)$:

i).

$$\varphi(f\xi) = f\varphi(\xi), \ \forall f \in C^\infty_{\mathcal{M}}\ , \ \xi \in \Upsilon_1^{\bullet,\bullet}.$$

ii).

$$\delta_2 \circ \varphi = \varphi \circ \delta_1,$$

$$L_Q \circ \varphi = \varphi \circ L_Q.$$

Denote by $\mathrm{Ch}(\mathbf{dg}{-}\mathbf{mod})$ the category of dg complexes over $(\mathcal{M}, Q)$.

# 6. dg complex

A dg complex $(\Upsilon^{\bullet}, L_Q, \delta)$ could be seen as a double complex:

$$(\oplus \Upsilon^{p,q}, L_Q, \delta).$$

The operation of taking total complex is a functor:

$$\mathrm{tot} : \mathrm{Ch}(\mathbf{dg}{-}\mathbf{mod}) \to \mathbf{dg}{-}\mathbf{mod},$$

$$(\Upsilon^{\bullet}, L_Q, \delta) \mapsto \left(\mathrm{tot}\Upsilon = \oplus_{p+q} \Upsilon^{p,q}\ ,\ L_Q^{\mathrm{tot}} = L_Q + \delta\right).$$

# 6. dg complex

Denote the category of dg complexes over $(\mathcal{M}, Q)$ by $Ch(\mathbf{dg}-\mathbf{mod})$.

A quasi-isomorphism $\varphi: (\Upsilon_1, L_Q, \delta_1) \to (\Upsilon_2, L_Q, \delta_2)$ of dg complexes is a morphism in $Ch(\mathbf{dg}-\mathbf{mod})$ that induces a quasi-isomorphism between the corresponding total complexes $(\mathrm{tot}\Upsilon_1, L_Q + \delta_1)$ and $(\mathrm{tot}\Upsilon_2, L_Q + \delta_2)$.

The derived category $D(\mathbf{dg}-\mathbf{mod})$ of dg complexes over $(\mathcal{M}, Q)$ is the Gabriel-Zisman localization of $Ch(\mathbf{dg}-\mathbf{mod})$ by the set of quasi-isomorphisms.

# 6. dg complex

Taking total complex can be regarded as a functor

$$\mathrm{tot}: \ Ch(\mathbf{dg}{-}\mathbf{mod}) \to \mathbf{dg}{-}\mathbf{mod}.$$

Denote by

$$\mathrm{tot}: \ D(\mathbf{dg}{-}\mathbf{mod}) \to \Pi(\mathbf{dg}{-}\mathbf{mod})$$

the induced functor between the two Gabriel-Zisman localizations of categories.

# 6. dg complex

A natural diagram summarizes the relations between all the categories that we introduced.

$$
\begin{array}{ccccc}
\mathbf{dg}-\mathbf{mod} & \longrightarrow & \mathrm{H}(\mathbf{dg}-\mathbf{mod}) & \longrightarrow & \Pi(\mathbf{dg}-\mathbf{mod}) \\
\uparrow {\scriptstyle \mathrm{tot}} & & & & \uparrow {\scriptstyle \mathrm{tot}} \\
Ch(\mathbf{dg}-\mathbf{mod}) & & \longrightarrow & & D(\mathbf{dg}-\mathbf{mod})
\end{array}
$$

# 6. dg complex of polyvector fields

$\otimes$: tensor product of dg modules.
$\widetilde{\otimes}$: tensor product of dg complexes.
$|\cdot|$: total degree of a dg complex.

dg module of polyvector fields:

$$\mathscr{T}^n_{poly} := (\Gamma(\widetilde{\wedge}^n T_{\mathcal{M}}), L_Q), \ \ n = 0, 1, \ldots$$

$$(\mathscr{T}^\bullet_{poly}, L_Q, 0) \in \mathrm{Ch}(\mathbf{dg}{-}\mathbf{mod}).$$

$$\mathscr{T}^{p,q}_{poly} = (\mathscr{T}^p_{poly})^q.$$

# 6. dg complex of polydifferential operators

dg modules of polydifferential operators:

$$\mathscr{D}^{n}_{\mathrm{poly}} := (D_{\mathcal{M}}^{\widetilde{\otimes} n} = \widetilde{\otimes}^{n} \mathscr{D}^{1}_{\mathrm{poly}}, L_Q),\ n = 0, 1, \ldots$$

dg complex of <u>polydifferential operators</u>:

$$(\mathscr{D}^{\bullet}_{\mathrm{poly}}, L_Q, d_H)$$

$d_H$ is the Hochschild coboundary:

$$\begin{gathered} d_H(D_1 \widetilde{\otimes} \cdots \widetilde{\otimes} D_n) = \\ (-1)^{\sum_{i=1}^{n} |D_i|}((-1)^{1+\sum_{i=1}^{n} |D_i|} 1 \widetilde{\otimes} D_1 \widetilde{\otimes} \cdots \widetilde{\otimes} D_n + \\ - \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i-1} |D_j|} D_1 \widetilde{\otimes} \cdots D_{i-1} \widetilde{\otimes} \Delta(D_i) \widetilde{\otimes} D_{i+1} \cdots D_n \\ + D_1 \widetilde{\otimes} \cdots \widetilde{\otimes} D_n \widetilde{\otimes} 1). \end{gathered}$$

We will consider $(\mathrm{tot}\mathscr{D}_{\mathrm{poly}}, L_Q + d_H) \in \mathbf{dg}{-}\mathbf{mod}$.

# 6. The dg Hopf algebra $\mathscr{D}_{\mathrm{poly}}$ of polydifferential operators

The space of polydifferential operators

$$\mathscr{D}_{\mathrm{poly}} := \oplus_{n\geqslant 0}\mathscr{D}^{n}_{\mathrm{poly}} = \oplus_{n\geqslant 0, m\in\mathbb{Z}}\mathscr{D}^{n,m}_{\mathrm{poly}}$$

admits a Hopf algebra structure:

- The multiplication:

$$\mathscr{D}_{\mathrm{poly}} \bigotimes \mathscr{D}_{\mathrm{poly}} \to \mathscr{D}_{\mathrm{poly}}$$

$$(D_1\widetilde{\otimes}\cdots\widetilde{\otimes}D_n)\bigotimes(D_{n+1}\widetilde{\otimes}\cdots\widetilde{\otimes}D_{n+m}) \mapsto D_1\widetilde{\otimes}\cdots\widetilde{\otimes}D_n\widetilde{\otimes}D_{n+1}\widetilde{\otimes}\cdots D_m.$$

- The comultiplication:

$$\mathscr{D}_{\mathrm{poly}} \to \mathscr{D}_{\mathrm{poly}} \bigotimes \mathscr{D}_{\mathrm{poly}},$$

$$D_1\widetilde{\otimes}\ldots D_n \mapsto \sum_{p+q=n}\ \sum_{(p,q)\text{-shuffle }\sigma} \kappa(\sigma)\ D_{\sigma(1)}\widetilde{\otimes}\ldots D_{\sigma(p)} \bigotimes D_{\sigma(p+1)}\widetilde{\otimes}\ldots D_{\sigma(n)}.$$

# 6. The dg Hopf algebra $\mathscr{D}_{\rm poly}$ of polydifferential operators

- The unit is the natural inclusion $\eta : C^\infty_{\mathcal{M}} = \mathscr{D}^0_{\rm poly} \hookrightarrow \mathscr{D}_{\rm poly}$.
- The counit $\varepsilon : \mathscr{D}_{\rm poly} \twoheadrightarrow \mathscr{D}^0_{\rm poly} = C^\infty_{\mathcal{M}}$ is the natural projection.
- The antipode is the map

$$t : \mathscr{D}_{\rm poly} \to \mathscr{D}_{\rm poly}$$

$$t(D_1 \widetilde{\otimes} D_2 \cdots \widetilde{\otimes} D_n) = (-1)^{\natural} D_n \widetilde{\otimes} \cdots D_2 \widetilde{\otimes} D_1,$$

where $\natural = \sum_{i=0}^{n-1} |D_{n-i}|(|D_1| + \cdots + |D_{n-i-1}|)$.

# 6. The dg Hopf algebra $\mathscr{D}_{\mathrm{poly}}$ of polydifferential operators

From the dg complex of polydifferential operators:

$$(\mathscr{D}^{\bullet}_{\mathrm{poly}}, L_Q, d_H),$$

we get a dg module $(\mathrm{tot}\mathscr{D}_{\mathrm{poly}}, L_Q + d_H)$, a Hopf algebra object, in the homotopy category $\Pi(\mathbf{dg}{-}\mathbf{mod})$.

# 6. The dg Hopf algebra $\mathscr{D}_{\mathrm{poly}}$ of polydifferential operators

The Lie bracket of two elements $D \in \mathscr{D}^i_{\mathrm{poly}}$ and $E \in \mathscr{D}^j_{\mathrm{poly}}$ is the element

$$[\![D, E]\!] = D \widetilde{\otimes} E - (-1)^{|D||E|} E \widetilde{\otimes} D \in \mathscr{D}^{i+j}_{\mathrm{poly}}.$$

Denote by $L(\mathscr{D}^1_{\mathrm{poly}})$ the smallest Lie subalgebra of $\mathscr{D}^\bullet_{\mathrm{poly}}$ containing $\mathscr{D}^1_{\mathrm{poly}}$. The space $L(\mathscr{D}^1_{\mathrm{poly}})$ is made of all $\mathbb{K}$-linear combinations of elements of the form $[\![D_1, \cdots, [\![D_{n-1}, D_n]\!], \cdots]\!]$ with $D_1, \cdots, D_n \in \mathscr{D}^1_{\mathrm{poly}}$.

# 6. The dg Hopf algebra $\mathscr{D}_{\mathrm{poly}}$ of polydifferential operators

We will need $(\mathrm{tot} L(\mathscr{D}^1_{\mathrm{poly}}), L_Q + d_H; [\![\ ,\ ]\!]) =$

free Lie algebra object spanned by $\mathrm{tot}\mathscr{D}^1_{\mathrm{poly}} = \mathcal{D}_{\mathcal{M}}[-1]$,

in $\mathbf{dg}-\mathbf{mod}$, the category of dg modules over $(\mathcal{M}, Q)$.

# 7. Atiyah class: connection

A smooth connection $\nabla$ on a dg module $(\mathfrak{N}, L_Q)$ is a $\mathbb{K}$-linear and degree 0 morphism:

$$\nabla : \mathscr{X}(\mathcal{M}) \otimes_{\mathbb{K}} \mathfrak{N} \to \mathfrak{N}$$

s.t.

$$\nabla_{fX}\xi = f\nabla_X\xi$$

$$\nabla_X(f\xi) = X(f)\xi + (-1)^{\widetilde{X}\widetilde{f}} f\nabla_X\xi$$

$\forall f \in C^\infty_{\mathcal{M}}$, $X \in \mathscr{X}(\mathcal{M})$, $\xi \in \mathfrak{N}$.

# 7. Atiyah class: Atiyah cocycle

The Atiyah cocyle

$$\alpha_{\mathfrak{N}}^{\nabla} := [L_Q, \nabla] \in \mathrm{Hom}^1(\mathscr{X}(\mathcal{M}) \otimes_{C^\infty_{\mathcal{M}}} \mathfrak{N}, \mathfrak{N}),$$

$$\alpha_{\mathfrak{N}}^{\nabla}(X, \xi) = L_Q(\nabla_X \xi) - \nabla_{[Q,X]}\xi - (-1)^{\tilde{X}} \nabla_X L_Q(\xi).$$

The <u>Atiyah class of the dg module</u> $\mathfrak{N}$

$$\alpha_{\mathfrak{N}} := [\alpha_{\mathfrak{N}}^{\nabla}] \in H^1\left(\mathrm{Hom}(\mathscr{X}(\mathcal{M}) \otimes_{C^\infty_{\mathcal{M}}} \mathfrak{N}, \mathfrak{N}), L_Q\right)$$

$$\simeq \mathrm{Hom}_{\mathrm{H}(\mathbf{dg-mod})}(\mathscr{X}(\mathcal{M})[-1] \otimes_{C^\infty_{\mathcal{M}}} \mathfrak{N}, \mathfrak{N}).$$

It is independent of $\nabla$. It can be regarded as a morphism

$$\alpha_{\mathfrak{N}} : (\mathscr{X}(\mathcal{M})[-1], L_Q) \otimes_{C^\infty_{\mathcal{M}}} (\mathfrak{N}, L_Q) \to (\mathfrak{N}, L_Q)$$

in the homology category $\mathrm{H}(\mathbf{dg-mod})$.

# 7. Atiyah class

The Atiyah class $\alpha_{\mathcal{M}}$ of a dg manifold $(\mathcal{M}, Q)$ is defined to be the Atiyah class of the dg module $(\mathscr{X}(\mathcal{M})[-1], L_Q)$,

which can be seen as a morphism

$$\alpha_{\mathcal{M}} : (\mathscr{X}(\mathcal{M})[-1], L_Q) \otimes_{C^\infty_{\mathcal{M}}} (\mathscr{X}(\mathcal{M})[-1], L_Q) \to (\mathscr{X}(\mathcal{M})[-1], L_Q)$$

in the homology category $\mathrm{H}(\mathbf{dg}-\mathbf{mod})$.

# 7. Atiyah class

**Theorem (Mehta-Stiénon-Xu 2015)**

*Given a torsion free connection $\nabla$ on $T_{\mathcal{M}}$, there is a $L_\infty$-algebra structure $\{\lambda_k\}_{k\geq 1}$ on $\mathscr{X}(\mathcal{M})[-1]$, s,t. $\lambda_1 = L_Q$, $\lambda_2 = \alpha^{\nabla}_{\mathcal{M}}$.*

As a corollary,

$$\alpha_{\mathcal{M}} : (\mathscr{X}(\mathcal{M})[-1], L_Q) \otimes_{C^\infty_{\mathcal{M}}} (\mathscr{X}(\mathcal{M})[-1], L_Q) \to (\mathscr{X}(\mathcal{M})[-1], L_Q)$$

defines a Lie algebra object in $\mathrm{H}(\mathbf{dg}-\mathbf{mod})$, and thus in the homotopy category $\Pi(\mathbf{dg}-\mathbf{mod})$.

Question: Does the universal enveloping algebra of the Lie algebra $(\mathscr{X}(\mathcal{M})[-1], L_Q; \alpha_{\mathcal{M}})$ exists in $\Pi(\mathbf{dg}-\mathbf{mod})$?

# 4. Main Results

If it exists, the universal enveloping algebra of a Lie algebra object $G$ in $Ch(\mathbf{dg}{-}\mathbf{mod})$ (resp. $D(\mathbf{dg}{-}\mathbf{mod})$) is an associative algebra object $U(G)$ in $Ch(\mathbf{dg}{-}\mathbf{mod})$ (resp. $D(\mathbf{dg}{-}\mathbf{mod})$) together with a morphism of Lie algebra objects $i: \ G \to U(G)$ satisfying the following universal property:
given any associative algebra object $K$ and any morphism of Lie algebras $f: \ G \to K$ in $Ch(\mathbf{dg}{-}\mathbf{mod})$ (resp. $D(\mathbf{dg}{-}\mathbf{mod})$), there exists a unique morphism of associative algebras $f': \ U(G) \to K$ in $Ch(\mathbf{dg}{-}\mathbf{mod})$ (resp. $D(\mathbf{dg}{-}\mathbf{mod})$) such that $f = f' \circ i$.

If exists, $U(G)$ is unique up to isomorphism in $Ch(\mathbf{dg}{-}\mathbf{mod})$ (resp. $D(\mathbf{dg}{-}\mathbf{mod})$).

# 8. Main results

Our results could be seen as a universal realization of Atiyah classes.

Theorem (Cheng-Chen-Ni)

i). *The natural inclusion map*

$$\theta : \ (\mathscr{X}(\mathcal{M})[-1], L_Q; \alpha_{\mathcal{M}}) \to (\mathrm{tot}\, L(\mathscr{D}^1_{\mathrm{poly}}), L_Q + d_H; [\![\ ,\ ]\!])$$

*is an isomorphism of Lie algebra objects in* $\Pi(\mathbf{dg}-\mathbf{mod})$*:*

$$\begin{array}{ccc}
(\mathscr{X}(\mathcal{M})[-1], L_Q) \otimes_{C^\infty_{\mathcal{M}}} (\mathscr{X}(\mathcal{M})[-1], L_Q) & \xrightarrow{\alpha_{\mathcal{M}}} & (\mathscr{X}(\mathcal{M})[-1], L_Q) \\
\downarrow{\scriptstyle \theta\otimes\theta} & & \downarrow{\scriptstyle \theta} \\
(\mathrm{tot}\, L(\mathscr{D}^1_{\mathrm{poly}}), L_Q + d_H) \otimes_{C^\infty_{\mathcal{M}}} (\mathrm{tot}\, L(\mathscr{D}^1_{\mathrm{poly}}), L_Q + d_H) & \xrightarrow{[\![\ ,\ ]\!]} & (\mathrm{tot}\, L(\mathscr{D}^1_{\mathrm{poly}}), L_Q + d_H).
\end{array}$$

# 4. Main Results

ii). The universal enveloping algebra

$$U((\mathscr{X}(\mathcal{M})[-1], L_Q; \alpha_{\mathcal{M}})) \simeq (\mathrm{tot}\mathscr{D}_{\mathrm{poly}}, L_Q + d_H),$$

which is a Hopf algebra object, in $\Pi(\mathbf{dg}-\mathbf{mod})$.

# 8. Main results

Our proof is based on

i). Poincaré-Birkhoff-Witt isomorphism in Lie theory,

ii). Hochschild-Kostant-Rosenberg quasi-isomorphism for dg manifolds (Liao-Stiénon-Xu 2017),

iii). Properties of Atiyah classes; hard calculations.

# 5. Application

Our main results could be adapted to the Atiyah class of a dg Lie algebroid, the proof is essentially the same.

Example: Fedosov dg Lie algebroid associated with Lie pairs.

# 5. Application: Lie pair

$M$: smooth manifold, $R := C^\infty(M, \mathbb{K})$.

A Lie algebroid $L = (L, [\ ,\ ], \rho)$:

i). $\mathbb{K}$-linear anchor map $\rho:\ L \to T_M$.

ii). Bracket $[\ ,\ ]:\ \Gamma L \otimes_{\mathbb{K}} \Gamma L \to \Gamma L$, s.t.

$$[X,\ fY] = f[X,\ Y] + (\rho(X)f)Y,$$

for all $X, Y \in \Gamma(L)$ and $f \in R$.

# 5. Application: Atiyah class of Lie pair

Lie pair $(L, A)$: a pair of Lie algebroids, where $A \subset L$ is a Lie sub-algebroid.

Let $j: B \to L$ be a splitting of the exact sequence

$$0 \to A \xrightarrow{i} L \xrightarrow{pr} B \to 0,$$

then $L \simeq A \oplus B$.

$A$-module structure on $B$ (Bott connection):

$$\Gamma(A) \otimes_{\mathbb{K}} \Gamma(B) \to \Gamma(B),$$
$$\nabla_a^{\mathrm{Bott}} b := pr([a, j(b)]),$$

for $a \in \Gamma(A), b \in \Gamma(B)$.

# 5. Application: Atiyah class of Lie pair

Let $\nabla : \Gamma(L) \otimes_{\mathbb{K}} \Gamma(B) \to \Gamma(B)$ be a torsion free $L$-connection.

The Atiyah cocycle $\alpha_B^{\nabla} \in \Gamma(A^{\vee} \otimes B^{\vee} \otimes End(B))$,

$$\alpha_B^{\nabla}(a, b)e := \nabla_a^{\mathrm{Bott}} \nabla_{j(b)} e - \nabla_{j(b)} \nabla_a^{\mathrm{Bott}} e - \nabla_{[a, j(b)]} e,$$

for all $a \in \Gamma(A)$, and $b, e \in \Gamma(B)$.

The Atiyah class of the Lie pair $(L, A)$ is the cohomology class

$$\alpha_B = [\alpha_B^{\nabla}] \in H^1_{\mathrm{CE}}(A, B^{\vee} \otimes End(B)).$$

# 5. Application: Atiyah class of Lie pair

$(A[1], d_{\mathrm{CE}}^A)$ is a dg manifold, $\Omega(A) := C^\infty(A[1])$.

Let $(B^!, d_{\mathrm{CE}}^{B^!})$ be the dg vector bundle over $(A[1], d_{\mathrm{CE}}^A)$ which is the pull back of $B$:

$$\begin{array}{ccc} (B^!, d_{\mathrm{CE}}^{B^!}) & \longrightarrow & B \\ \downarrow & & \downarrow \\ (A[1], d_{\mathrm{CE}}^A) & \longrightarrow & M\,. \end{array}$$

# 5. Application: Atiyah class of Lie pair

The Lie pair Atiyah class can be regarded as a Lie bracket

$$\alpha_B : (\Gamma(B^!)[-1], d_{\mathrm{CE}}^{B^!}) \otimes_{\Omega(A)} (\Gamma(B^!)[-1], d_{\mathrm{CE}}^{B^!}) \to (\Gamma(B^!)[-1], d_{\mathrm{CE}}^{B^!})$$

in the homology category $\mathrm{H}(\Omega(A)-\mathbf{mod})$ of dg modules over $\Omega(A)$.

Equivalently, it can be regarded as a Lie bracket

$$\alpha_B : \Gamma(B)[-1] \otimes_{C^\infty_M} \Gamma(B)[-1] \to \Gamma(B)[-1]$$

in the derived category $D^b(\mathcal{A})$ of $A$-modules (Chen-Stiénon-Xu 2014).

# 5. Application: Fedosov dg Lie algebroid

Hochschild complex $(D^\bullet_{\mathrm{poly}}(B), d_H)$ associated with Lie pair $(L, A)$, where $D^n_{\mathrm{poly}}(B) = \otimes^n_{C^\infty_M} \frac{U(L)}{U(L)\Gamma(A)}$ (Chen-Stiénon-Xu 2014).

Fedosov dg manifold associated with $(L, A)$ (Stiénon-Xu 2016, Batakidis-Voglaire 2018):

$$(\mathcal{M}, Q) := (L[1] \oplus B, d_L^{\nabla^\natural}).$$

$$C^\infty_{\mathcal{M}} = \Gamma(SB^\vee \otimes \wedge^\bullet L) = \Gamma(SB^\vee) \otimes_{C^\infty_M} \Omega(L).$$

# 5. Application: Fedosov dg Lie algebroid

Construction:

i). PBW isomorphism (Laurent-Gengoux-Stiénon-Xu 2012):

$$\mathrm{pbw}^{\nabla,j}: \ \Gamma(SB) \to D^1_{\mathrm{poly}}(B),$$

ii). canonical connection:

$$\nabla^{\mathrm{can}}: \ \Gamma(L) \otimes_{\mathbb{K}} D^1_{\mathrm{poly}}(B) \to D^1_{\mathrm{poly}}(B),$$
$$\nabla^{\mathrm{can}}_l u := l \cdot u,$$

iii). the $L$-connection $\nabla^{\lightning}: \Gamma(L) \otimes_{\mathbb{K}} \Gamma(SB) \to \Gamma(SB)$ is the pull back of $\nabla^{\mathrm{can}}$ via $\mathrm{pbw}^{\nabla,j}$.

$d_L^{\nabla^{\lightning}}$ is the Chevalley-Eilenberg differential associated with $\nabla^{\lightning}$.

# 5. Application: Fedosov dg Lie algebroid

$\mathcal{M} = L[1] \oplus B.$

Natural maps:
$(A[1], d_{\mathrm{CE}}^{A}) \xrightarrow{\iota} (\mathcal{M} = L[1] \oplus B, Q = \nabla^{\natural}) \xrightarrow{\pi} (L[1], d_{\mathrm{CE}}^{L}).$

<u>Fedosov dg Lie algebroid</u>

$$(\mathcal{F}, L_Q) := ker\, \pi_*$$

.

# 5. Application: Fedosov dg Lie algebroid

The diagram formed by natural maps

$$\begin{array}{ccc} (B^!, d_{\mathrm{CE}}^{B^!}) & \xrightarrow{\iota} & (\mathcal{F}, L_Q) \\ \downarrow & & \downarrow \\ (A[1], d_{\mathrm{CE}}^{A}) & \xrightarrow{\iota} & (\mathcal{M}, Q) \end{array}$$

is a pull back diagram of dg vector bundles. In other words, $\iota^*(\mathcal{F}, L_Q) = (B^!, d_{\mathrm{CE}}^{B^!})$ or

$$(\Gamma(B^!), d_{\mathrm{CE}}^{B^!}) = (\Gamma(\mathcal{F}), L_Q) \otimes_{C^\infty_{\mathcal{M}}} (\Omega(A), d_{\mathrm{CE}}^{A}).$$

# 5. Application: Fedosov dg Lie algebroid

The following fact is due to Liao-Stiénon-Xu 2019.

The restriction map $\iota^* : (\Gamma(\mathcal{F}), L_Q) \to (\Gamma(B^!), d_{\mathrm{CE}}^{B^!})$ is a quasi-isomorphism of dg modules.

Consequently, the following diagram commutes in the homotopy category $\Pi(\mathbf{dg}{-}\mathbf{mod})$:

$$\begin{array}{ccc} (\Gamma(\mathcal{F})[-1], L_Q) \otimes_{C^\infty_{\mathcal{M}}} (\Gamma(\mathcal{F})[-1], L_Q) & \xrightarrow{\alpha_{\mathcal{F}}} & (\Gamma(\mathcal{F})[-1], L_Q) \\ \downarrow{\scriptstyle \iota^* \otimes \iota^*} & & \downarrow{\scriptstyle \iota^*} \\ (\Gamma(B^!)[-1], d_{\mathrm{CE}}^{B^!}) \otimes_{\Omega(A)} (\Gamma(B^!)[-1], d_{\mathrm{CE}}^{B^!}) & \xrightarrow{\alpha_B} & (\Gamma(B^!)[-1], d_{\mathrm{CE}}^{B^!}) . \end{array}$$

# 5. Application: Fedosov dg Lie algebroid

Define a map $I$ of dg modules over $(\mathcal{M}, Q)$ —

$$\begin{array}{ccc} (\mathrm{tot}\,\mathscr{D}_{\mathrm{poly}}(\mathcal{F}), L_Q + d_H) & \xrightarrow{\mathrm{pbw}^{\nabla,j}\otimes_{C^\infty_M}\mathrm{id}} & (D^\bullet_{\mathrm{poly}}(B)\otimes_{C^\infty_M} C^\infty_{\mathcal{M}}, d_{\mathrm{can}} + d_H\otimes_{C^\infty_M}\mathrm{id}) \\ & \searrow^{I} & \Big\downarrow{\scriptstyle \mathrm{id}\otimes_{C^\infty_M}\iota^*} \\ & & (D^\bullet_{\mathrm{poly}}(B)\otimes_{C^\infty_M}\Omega(A), d^B_{\mathrm{CE}} + d_H\otimes_{C^\infty_M}\mathrm{id})\,. \end{array}$$

The following facts are due to Bandiera-Stiénon-Xu 2019:

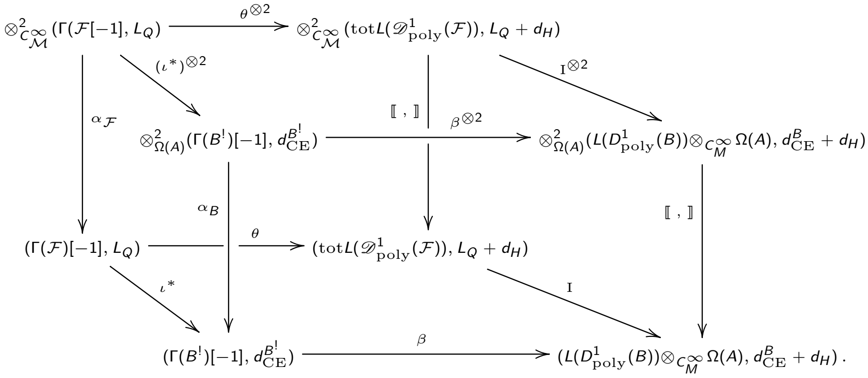
- the horizontal map $\mathrm{pbw}^{\nabla,j}\otimes_{C^\infty_M}\mathrm{id}$ is an isomorphism of dg modules;
- the vertical map $\mathrm{id}\otimes_{C^\infty_M}\iota^*$ is a quasi-isomorphism of dg modules.

Thus $I$ is a quasi-isomorphism of dg modules.

# 5. Application: big diagram

## Theorem (Cheng-Chen-Ni)

*The following diagram commutes in the homotopy category* $\Pi(\mathbf{dg}\!-\!\mathbf{mod})$*:*




*Moreover, the map* $\beta$ *in the front lower edge is an isomorphism in* $\Pi(\mathbf{dg}\!-\!\mathbf{mod})$.

# 5. Application: big diagram

Thus we can recover:

## Theorem (Chen-Stiénon-Xu 2014)

i). *The inclusion map* $\beta: \Gamma(B[-1]) \to L(D^1_{\mathrm{poly}}(B))$ *is an isomorphism of Lie algebra objects in the derived category* $D^b(\mathcal{A})$ *of A-modules,*

$$\begin{array}{ccc} \Gamma(B[-1]) \otimes_{C^\infty_M} \Gamma(B[-1]) & \xrightarrow{\beta\otimes\beta} & L(D^1_{\mathrm{poly}}(B)) \otimes_{C^\infty_M} L(D^1_{\mathrm{poly}}(B)) \\ \downarrow{\scriptstyle \alpha_B} & & \downarrow{\scriptstyle [\![\, , \,]\!]} \\ \Gamma(B[-1]) & \xrightarrow{\beta} & L(D^1_{\mathrm{poly}}(B)) \,. \end{array}$$

$(L(D^1_{\mathrm{poly}}(B); [\![\, , \,]\!]) =$ free Lie algebra spanned by $D^1_{\mathrm{poly}}(B)$.

# 5. Application: big diagram

ii). The universal enveloping algebra

$$U((\Gamma(B[-1]); \alpha_B)) \simeq (D^{\bullet}_{\mathrm{poly}}(B), d_H),$$

which is a Hopf algebra, in $D^b(\mathcal{A})$.

Hopf algebras arising from dg manifolds, available at https://arxiv.org/abs/1911.01388

Thank You !