Atiyah classes for generalized holomorphic vector bundles

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Plans:

1. Atiyah classes for holomorphic vector bundles
2. Generalized complex manifolds and generalized holomorphic functions
3. Generalized holomorphic vector bundles
4. Atiyah classes

Atiyah classes for holomorphic vector bundles

Atiyah class: obstruction of the existence of holomorphic connections on a holomorphic vector bundle.

Holomorphic vector bundles

Definition

Let M be a complex manifold. A **holomorphic vector bundle** of rank r on M is a complex manifold E with a holomorphic map

$$\pi : E \to M$$

and a dim r complex v.s. str. on $E_x = \pi^{-1}(x)$ satisfying: There exists an open covering $\{U_i\}$ of M and holomorphic homeomorphisms: $\varphi_i : \pi^{-1}(U_i) \cong U_i \times \mathbb{C}^r$ commuting with the projections to U_i such that the induced map $\pi^{-1}(x) \cong \mathbb{C}^r$ is $\mathbb{C}$-linear.

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$\Rightarrow$ Let $E|_{U_i} \cong U_i \times \mathbb{C}^r$ and let $\{e_i\}$ be a basis of $\Gamma(E|_{U_i})$. For $s = \sum_\lambda s_\lambda e_\lambda$ with $s_\lambda \in C^\infty(U_i)$, define

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$$\bar\partial : \Gamma(E) \to \Gamma((T^{0,1}M)^* \otimes E), \qquad s \mapsto \sum_i \bar\partial(s_\lambda)e_\lambda.$$

$\Leftarrow$ if there exists $D^{0,1} : \Gamma(E) \to \Gamma((T^{0,1}M)^* \otimes E)$ such that $(D^{0,1})^2 = 0$, then there is a unique holomorphic vector bundle str. on E such that $D^{0,1} = \bar\partial$.

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Let E be a holomorphic bundle on a complex manifold M. A **holomorphic connection** on E is a $\mathbb{C}$-linear map (of sheaves) $D : E \to \Omega_M \otimes E$ with

$$D(fs) = \partial(f) \otimes s + fD(s),$$

for any local holomorphic function f of M and any local holomorphic section s of E.

Here E and Ω_M denote the sheaves of holomorphic sections of E and $(T^{1,0}M)^*$.

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- D sends holomorphic sections of E to holomorphic sections of $(T^{1,0}M)^* \otimes E$. Or, for any holomorphic tangent vector field X, D_X preserves the holomorphic sections of E.

Let E be a holomorphic bundle and let $\{U_i\}$ be an open covering s.t. there exist holomorphic trivializations $\varphi_i : E|_{U_i} \cong U_i \times \mathbb{C}^r$.
A local holomorphic connection on $U_i \times \mathbb{C}^r$ is $\partial + A_i$, where A_i is a matrix valued holomorphic 1-form on U_i. They can be glued to a connection on E iff

$$\varphi_i^{-1} \circ (\partial + A_i) \circ \varphi_i = \varphi_j^{-1} \circ (\partial + A_j) \circ \varphi_j$$

on $U_{ij} = U_i \cap U_j$, equivalently,

$$\varphi_j^{-1} \circ (\varphi_{ij}^{-1} \circ \partial \circ \varphi_{ij} - \partial) \circ \varphi_j = \varphi_j^{-1} \circ A_j \circ \varphi_j - \varphi_i^{-1} \circ A_i \circ \varphi_i,$$

where $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$.
By the relation $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = 1$, the left hand side is actually a Čech cocycle.

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Definition

The Atiyah class

$$A(E) \in \mathrm{H}^1(M, \Omega_M \otimes \mathrm{End}(E))$$

of the holomorphic bundle E is given by the Čech cocycle

$$A(E) = \{U_{ij}, \varphi_j^{-1} \circ (\varphi_{ij}^{-1} d(\varphi_{ij})) \circ \varphi_j\}.$$

Proposition (Atiyah)

A holomorphic bundle E admits a holomorphic connection iff its Atiyah class $A(E) \in \mathrm{H}^1(M, \Omega_M \otimes \mathrm{End}(E))$ is trivial.

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$A(E)$ is related with the curvature of E.

Let (E, h) be a hermitian holomorphic vector bundle. A connection ∇ on E is called a **Chern connection** if

$$\nabla h = 0, \qquad \nabla^{0,1} = \bar{\partial},$$

where $\nabla = \nabla^{1,0} + \nabla^{0,1} : \Gamma(E) \to \Gamma(((T^*M)^{1,0} \oplus (T^*M)^{0,1}) \otimes E)$.

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The curvature of a Chern connection $F_\nabla \in \Omega^{1,1}(M) \otimes \mathrm{End}(E)$ and the Bianchi identity yields

$$0 = (\nabla(F_\nabla))^{1,2} = \bar{\partial}(F_\nabla).$$

This gives rise to a Dolbeault cohomology class $[F_\nabla]$.

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Proposition

For the curvature F_∇ of the Chern connection on a hermitian holomorphic vector bundle (E, h) one has

$$[F_\nabla] = A(E) \in \mathrm{H}^1(M, \Omega_M \otimes \mathrm{End}(E)).$$

Let E be a holomorphic vector bundle on a complex manifold M. Let $\mathfrak{J}^1 E$ the vector bundle of the first jets of holomorphic sections of E. It fits into the short exact sequence

$$0 \to \Omega_M \otimes E \to \mathfrak{J}^1 E \to E \to 0$$

of holomorphic vector bundles.
The **Atiyah class** of E is the extension class

$$\alpha_E \in \mathrm{Ext}^1_M(E, \Omega_M \otimes E) \cong \mathrm{H}^1(M, \Omega_M \otimes \mathrm{End}(E)).$$

Generalized complex manifolds and generalized holomorphic functions

Let M be a smooth manifold. On $TM \oplus T^*M$, there is a canonical bilinear form valued in $C^\infty(M)$:

$$(X + \xi, Y + \eta) = \xi(Y) + \eta(X), \qquad X, Y \in \mathfrak{X}(M), \xi, \eta \in \Omega^1(M),$$

and a skew-symmetric bracket, called the **Courant bracket**:

$$[X + \xi, Y + \eta] := [X, Y] + \mathcal{L}_X \eta - \mathcal{L}_Y \xi - \frac{1}{2} d(\eta(X) - \xi(Y)).$$

Also we have the anchor $\rho : TM \oplus T^*M \to TM$, the projection.

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Properties

- $[[u,v],w] + c.p. = \frac{1}{6}d(([u,v],w) + ([v,w],u) + ([w,u],v))$;
- $[u, fv] = f[u,v] + \rho(u)fv - (u,v)df$;
- $\rho(u)(v,w) = ([u,v] + d(u,v), w) + (v, [u,w] + d(u,w))$,

for $u, v, w \in \Gamma(TM \oplus T^*M)$.

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—— Courant algebroid $(\mathcal{C}, (\cdot,\cdot), [\cdot,\cdot], \rho)$ Liu-Xu-Weinstein 97.

Definition (Hitchin, Gualtieri)

A **generalized complex structure** on M is an endomorphism $\mathcal{J}$ of $TM \oplus T^*M$ such that

- $\mathcal{J}^2 = -1$;
- $(\mathcal{J}u, \mathcal{J}v) = (u, v)$;
- the $+i$-eigenbundle $L \subset (TM \oplus T^*M) \otimes \mathbb{C}$ of $\mathcal{J}$ is closed under the Courant bracket (**integrability condition).**

In block-diagonal form, a skew-adjoint transformation on $TM \oplus T^*M$ is

$$\mathcal{J} = \begin{pmatrix} J & \beta \\ B & -J^* \end{pmatrix},$$

where $J \in \mathrm{End}(TM)$ and $B \in \Omega^2(M)$ and $\beta \in \mathfrak{X}^2(M)$.

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A **B-field transform** is an automorphism of $TM \oplus T^*M$ given by a 2-form $B \in \Omega^2(M)$ via:

$$X + \xi \mapsto X + \xi + \iota_X B, \qquad X \in \mathfrak{X}(M), \xi \in \Omega^1(M).$$

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A key feature of generalized complex geometry is that its symmetry group is $\mathrm{Diff}(M) \ltimes \Omega^2_{cl}(M)$:

$$(Bu, Bv) = (u, v), \qquad [Bu, Bv] = B[u, v], \qquad B \in \Omega^2_{cl}(M).$$

With respect to the Lie algebroids L and L_-, the $+i$ and $-i$-eigenbundles of a generalized complex structure, we have two Lie algebroid differentials

$$
\begin{aligned}
d_+ &: \Gamma(\wedge^\bullet L^*) \to \Gamma(\wedge^{\bullet+1} L^*),\\
d_- &: \Gamma(\wedge^\bullet L_-^*) \to \Gamma(\wedge^{\bullet+1} L_-^*).
\end{aligned}
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$(TM \oplus T^*M) \otimes \mathbb{C} = L \oplus L_-$. Unlike $d = \partial + \bar{\partial}$, no $d_+ + d_-$.

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$(TM \oplus T^*M) \otimes \mathbb{C} = L \oplus L_-$. Unlike $d = \partial + \bar{\partial}$, no $d_+ + d_-$.

Definition

A function $f \in C^\infty(M)$ on a generalized complex manifold $(M, \mathcal{J})$ is called a **generalized holomorphic function** if it satisfies $d_- f = 0$.

This definition is invariant under the B-transform. Under a B-transform,

$$L_- \mapsto B(L_-), \qquad d_- \mapsto d_-^B = d_- + B,$$

so if $d_- f = (X, \xi) = 0$,

$$d_-^B f = (X, \xi + B(X)) = 0.$$

Example

Let J be a complex structure on M. The endomorphism of $TM \oplus T^*M$

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}$$

is a generalized complex structure on M. Its $+i$-eigenbundle

$$L_J = T^{1,0}M \oplus (T^*M)^{0,1}$$

is integrable iff J is a complex structure. Moreover, we have

$$d_- = \bar{\partial}.$$

Thus $f \in C^\infty(M)$ is a generalized holomorphic function if it is a holomorphic function.

Example

Consider the endomorphism

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix},$$

where ω is a symplectic structure on M. The $+i$-eigenbundle

$$L_\omega = \{X - i\omega(X) | X \in T_{\mathbb{C}}M\}.$$

is integrable iff $d\omega = 0$. In this case,

$$d_- = d.$$

So a function $f \in C^\infty(M)$ is generalized holomorphic if it is constant.

Example

Let M be a complex manifold with a complex structure J. If there is a bivector field β on M such that

$$\mathcal{J} = \begin{pmatrix} J & \beta \\ 0 & -J^* \end{pmatrix}$$

is a generalized complex structure on M, then $\pi = J \circ \beta + i\beta$ is a holomorphic Poisson structure on M, i.e.

$$\pi \in \Gamma(\wedge^2 T^{1,0} M), \qquad \bar{\partial}\pi = 0, \qquad [\pi, \pi] = 0.$$

The $+i$-eigenbundle is

$$L = \{Y + \frac{\beta(\eta)}{2i} + \eta | Y \in T^{1,0} M, \eta \in (T^{0,1} M)^*\}.$$

In this case,

$$d_- = \bar{\partial} - \frac{1}{4}[\pi, \cdot].$$

Hence f is generalized holomorphic if it is a holomorphic Casimir function.

(**Generalized Darboux Theorem**) Any regular point in a generalized complex manifold has a neighborhood which is equivalent, up to a diffeomorphism and a B-transform, to a product of an open set in $\mathbb{C}^k$ with an open set in the standard symplectic space $(\mathbb{R}^{2n-2k}, \omega_0)$, i.e.

$$(U_p, \mathcal{J}) \cong (V \times W, e^B(\mathcal{J}_{J_0} \times \mathcal{J}_{\omega_0})e^{-B}).$$

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Choose the generalized Darboux coordinates (z, p, q). A function $f : M \to \mathbb{C}$ is generalized holomorphic iff

$$\frac{\partial f}{\partial \bar{z}_\lambda} = 0, \qquad \frac{\partial f}{\partial p_\mu} = \frac{\partial f}{\partial q_\mu} = 0, \qquad \lambda = 1, \cdots, k; \mu = 1, \cdots, n-k.$$

A **generalized holomorphic homeomorphism** $f : (M, \mathcal{J}_M) \to (N, \mathcal{J}_N)$ is homeomorphism sastifying

$$\begin{pmatrix} f_* & 0 \\ 0 & (f^{-1})^* \end{pmatrix} \circ \mathcal{J}_M = \mathcal{J}_N \circ \begin{pmatrix} f_* & 0 \\ 0 & (f^{-1})^* \end{pmatrix}.$$

Generalized holomorphic vector bundles

Definition (Jia-Lang-Liu)

Suppose that M is a generalized complex manifold. A real vector bundle $\pi : E \to M$ is called a **generalized holomorphic vector bundle**, if

(1) E is a generalized complex manifold;

(2) there is an open cover $\{U_i\}_{i\in I}$ of M and a family of local trivializations $\{\varphi_i : E|_{U_i} = \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r\}_{i\in I}$ satisfying that φ_i for each i is a generalized holomorphic homeomorphism, where $U_i \times \mathbb{C}^r$ is associated with the standard product generalized complex structure.

Proposition

Let E be a real vector bundle on M with a family of local trivializations $\{\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^{2r}\}$ and transition functions $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : U_i \cap U_j \to \mathrm{GL}(2r, \mathbb{R})$. Then E is generalized holomorphic vector bundle on M with the local trivialization $\{\varphi_i\}$ if and only if

(1) $\varphi_{ij}(p) \in \mathrm{GL}(r, \mathbb{C})$, so E is a complex vector bundle;

(2) each entry $A_{\lambda\mu} : U_i \cap U_j \to \mathbb{C}$ of $\varphi_{ij} = (A_{\lambda\mu})_{r\times r}$ is a generalized holomorphic function.

Sketch of proof

E is a GHVB with local trivialization $\{\varphi_i\}$ iff

$$\varphi_{ij} = \varphi_i \circ \varphi_j^{-1} : U_{ij} \times \mathbb{C}^r \to U_{ij} \times \mathbb{C}^r$$

is a generalized holomorphic homeomorphism for any fixed i, j. Namely,

$$\begin{pmatrix} (\varphi_{ij})_* & 0 \\ 0 & (\varphi_{ji})^* \end{pmatrix} \circ \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{pmatrix} = \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{pmatrix} \circ \begin{pmatrix} (\varphi_{ij})_* & 0 \\ 0 & (\varphi_{ji})^* \end{pmatrix}.$$

This guarantees that

$$\begin{pmatrix} (\varphi_i^{-1})_* & 0 \\ 0 & (\varphi_i)^* \end{pmatrix} \circ \begin{pmatrix} \mathcal{J}_{11} & \mathcal{J}_{12} \\ \mathcal{J}_{21} & \mathcal{J}_{22} \end{pmatrix} \circ \begin{pmatrix} (\varphi_i)_* & 0 \\ 0 & (\varphi_i^{-1})^* \end{pmatrix}$$

gives a generalized complex structure on $E|_{U_i} = \pi^{-1}(U_i)$, independent of φ_i.

Denote by $(\mathbb{C}^r, J_0)$ and let $\begin{pmatrix} J & \beta \\ B & -J^* \end{pmatrix}$ be the generalized complex structure (GCS) on M. Then the GCS $\mathcal{J}$ on $U_{ij} \times \mathbb{C}^r$ is expressed as

$$\mathcal{J}_{11} = \begin{pmatrix} J & 0 \\ 0 & J_0 \end{pmatrix}, \quad \mathcal{J}_{12} = \begin{pmatrix} \beta & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{J}_{21} = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{J}_{22} = \begin{pmatrix} -J^* & 0 \\ 0 & -J_0^* \end{pmatrix}.$$

Unraveling the above equation, we get

$$\begin{aligned} (\varphi_{ij})_{*(p,v)} \circ \mathcal{J}_{11} &= \mathcal{J}_{11} \circ (\varphi_{ij})_{*(p,v)}; \\ (\varphi_{ij})_{*(p,v)} \circ \mathcal{J}_{12} &= \mathcal{J}_{12} \circ (\varphi_{ji})^*_{(p,v)}; \\ (\varphi_{ji})^*_{(p,v)} \circ \mathcal{J}_{21} &= \mathcal{J}_{21} \circ (\varphi_{ij})_{*(p,v)}; \\ (\varphi_{ji})^*_{(p,v)} \circ \mathcal{J}_{22} &= \mathcal{J}_{22} \circ (\varphi_{ji})_{*(p,v)}. \end{aligned}$$

This implies (1) and (2).

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(1) A generalized holomorphic vector bundle (GHVB) on a complex manifold is a holomorphic vector bundle;

(2) A GHVB on a symplectic manifold is a complex vector bundle with a flat connection.

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Let M be a holomorphic Poisson manifold. A **Poisson module** is a locally free sheaf $\mathcal{O}(E)$ with an action $s \mapsto \{f, s\}$ of the structure sheaf with the properties

$$\{f, gs\} = \{f, g\}s + g\{f, s\}, \qquad \{\{f, g\}, s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\}.$$

(E is a Poisson module $\Longleftrightarrow T^*_\pi M$-module)

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(2) A GHVB on a symplectic manifold is a complex vector bundle with a flat connection.
Let M be a holomorphic Poisson manifold. A **Poisson module** is a locally free sheaf $\mathcal{O}(E)$ with an action $s \mapsto \{f, s\}$ of the structure sheaf with the properties

$$\{f, gs\} = \{f, g\}s + g\{f, s\}, \qquad \{\{f, g\}, s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\}.$$

(E is a Poisson module $\Longleftrightarrow T^*_\pi M$-module)

(3) A GHVB on a holomorphic Poisson manifold is a holomorphic bundle with a Poisson module structure given by

$$\{f, s\} := \sum_{\lambda=1}^{r} \{f, s_\lambda\}_M e_\lambda, \qquad s|_{U_i} = \sum_{\lambda=1}^{r} s_\lambda e_\lambda,$$

where $\{e_1, \cdots, e_r\}$ is a basis of $\Gamma(E|_{U_i})$.

Theorem

Let $(M, \mathcal{J})$ be a generalized complex manifold and let $L_- \subset T_{\mathbb{C}}M \oplus T^*_{\mathbb{C}}M$ be the $-i$-eigenbundle of $\mathcal{J}$. If E is a generalized holomorphic vector bundle on M, there exists an L_--connection $\overline{\partial}_L$ on E such that $\overline{\partial}_L^2 = 0$.

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To define

$$\overline{\partial}_L : \Gamma(E) \to \Gamma(L_-^* \otimes E),$$

let $\{e_1, \cdots, e_r\}$ be a basis of $\Gamma(E|_{U_i})$. For any $s|_{U_i} = \sum_{\lambda=1}^r s_\lambda e_\lambda \in \Gamma(E|_{U_i})$ with $s_\lambda \in C^\infty(U_i)$,

$$\overline{\partial}_L(s)|_{U_i} := \sum_{\lambda=1}^{r} (d_- s_\lambda) \otimes e_\lambda, \qquad (\bar{\partial} \mapsto d_-).$$

It is well-defined since the transition functions are generalized holomorphic.

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It is well-defined since the transition functions are generalized holomorphic.
Does it work the other way around?

Generalized holomorphic vector bundles in the literature

Definition (Gualtieri)

A generalized holomorphic bundle on a generalized complex manifold $(M, \mathcal{J})$ is a vector bundle E with an L_-**-module**, i.e., an operator $\bar{D} : \Gamma(E) \to \Gamma(L_-^* \otimes E)$ such that

$$\bar{D}(fs) = d_-(f)s + f\bar{D}s; \qquad \bar{D}^2 = 0$$

for $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

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In general, there is no generalized complex structure on the total space E, which must relate with the Poisson str. on M: $\{f, g\} = (d_+ f, d_- g)$.

- Under what conditions the total space E of a generalized holomorphic bundle admits a generalized complex str. such that $\bar{D} = \bar{\partial}_L$?
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The GCS on the product space needs to be discussed, which can be the deformation of the product GCSs on $U_i \times \mathbb{C}^r$.

Generalized holomorphic tangent and cotangent bundles

Proposition

Let $(M, \mathcal{J})$ be a regular generalized complex manifold. Then $G^*M := L_- \cap T^*_{\mathbb{C}}M$ is a generalized holomorphic vector bundle on M, which is called the **generalized holomorphic cotangent bundle** of M.

Generalized holomorphic tangent bundle $GM := L \cap T_{\mathbb{C}}M$.

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(1) When M is a complex manifold, we have $G^*M = (T^{1,0}M)^*$ and $GM = T^{1,0}M$;

(2) When M is a symplectic manifold, then G^*M is degenerated to a vector bundle of rank 0 on M and $GM = TM$.

(3) For a regular holomorphic Poisson manifold $(M, \mathcal{J})$, $GM = T^{1,0}M$ and $G^*M = \ker\pi^\sharp \cap (T^{1,0}M)^*$.

The regularity is not essential.

A section s of a generalized holomorphic vector bundle is called **generalized holomorphic** if $\overline{\partial}_L s = 0$.
Choosing a local trivialization $\varphi : E|_U \to U \times \mathbb{C}^r$, a section s can be written locally as

$$s = (s_1, \cdots, s_r), \qquad s_i : U \to \mathbb{C}.$$

Then s is a generalized holomorphic if all s_i are generalized holomorphic functions on M.

The first jet bundle of a generalized holomorphic bundle

Denote by $\Gamma_m(E)$ the space of local generalized holomorphic sections around m. Two local sections $\phi, \psi \in \Gamma_m(E)$ are **equivalent** iff

$$\phi(m) = \psi(m), \qquad \phi_{*m} = \psi_{*m}.$$

We denote the equivalence class of ϕ at m as $[\phi]_m$. Define

$$\mathfrak{J}^1 E = \{[\phi]_m | m \in M, \phi \in \Gamma_m(E)\}.$$

Locally, $\phi \sim \psi$ iff there exists a local coordinate system $(E|_{U_i}, \varphi_i; z, p, q, u^\alpha)$ such that

$$\frac{\partial u^\alpha \circ \phi}{\partial z_\lambda}|_m = \frac{\partial u^\alpha \circ \psi}{\partial z_\lambda}|_m, \qquad \alpha = 1, \cdots r; \lambda = 1, \cdots, k,$$

where $\varphi_i : E|_{U_i} \to U_i \times \mathbb{C}^r$ is a local trivialization of E, (z, p, q) is a coordinate system on U_i and $(u^\alpha)_{\alpha=1}^r$ is a coordinate system along the fiber.

Proposition

We have that $\mathfrak{J}^1 E$ is a generalized holomorphic bundle on M and it fits into the short exact sequence

$$0 \to G^* M \otimes E \to \mathfrak{J}^1 E \to E \to 0$$

of generalized holomorphic bundles on M.

Generalized holomorphic connections

Definition

Let E be a generalized holomorphic bundle on a generalized complex manifold M.
A **generalized holomorphic connection** on E is a complex linear map $D : E \to G^*M \otimes E$ (of sheaves) such that

$$D(fs) = d_+ f \otimes s + f D(s)$$

for all local generalized holomorphic function f on M and all local generalized holomorphic section s on E.

Here E and G^*M denotes the sheaves of generalized holomorphic sections of E and G^*M.
Since $d_+ f + d_- f = \rho^*(df) \in \Gamma(T^*_{\mathbb{C}} M)$, we have $d_+ f \in \Gamma(G^*M)$ and $d_-(d_+ f) = 0$.

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Lemma

Let E be a generalized holomorphic vector bundle on M. A complex linear map $D : E \to G^*M \otimes E$ is a generalized holomorphic connection on E iff D_X preserves generalized holomorphic sections of E, where X is a generalized holomorphic vector field.

1. When M is a complex manifold, a generalized holomorphic connection on E is a holomorphic connection;
2. When M is a symplectic manifold, since G^*M is of rank 0, the generalized holomorphic connection on E can only be zero ($d_+ = 0$). It is also clear from the first jet bundle.

Atiyah classes I

With respect to a trivialization $\varphi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ on E, we may write a local generalized holomorphic connection on $U_i \times \mathbb{C}^r$ in the form $d_+ + A_i$, where A_i is a matrix valued generalized holomorphic one-form on U_i. They can be glued to a connection on E iff

$$\varphi_i^{-1} \circ (d_+ + A_i) \circ \varphi_i = \varphi_j^{-1} \circ (d_+ + A_j) \circ \varphi_j$$

on U_{ij}, equivalently,

$$\varphi_j^{-1} \circ (\varphi_{ij}^{-1} \circ d_+ \circ \varphi_{ij} - d_+) \circ \varphi_j = \varphi_j^{-1} \circ A_j \circ \varphi_j - \varphi_i^{-1} \circ A_i \circ \varphi_i,$$

where $\varphi_{ij} = \varphi_i \circ \varphi_j^{-1}$. Also, by the relation $\varphi_{ij} \circ \varphi_{jk} \circ \varphi_{ki} = 1$, the left hand side of the above equation is actually a cocycle.

Definition

The **Atiyah class**

$$A(E) \in \mathrm{H}^1(M, G^*M \otimes \mathrm{End}(E))$$

of a generalized holomorphic vector bundle E on a generalized complex manifold $(M, \mathcal{J})$ is given by the Čech cocycle

$$A(E) = \{U_{ij}, \varphi_j^{-1} \circ (\varphi_{ij}^{-1} d_+(\varphi_{ij})) \circ \varphi_j\},$$

where d_+ is the Lie algebroid differential of the $+i$-eigenbundle L_+ of $\mathcal{J}$.

Theorem

Let E be a generalized holomorphic bundle on a generalized complex manifold M. Then E admits a generalized holomorphic connection iff the Atiyah class

$$A(E) \in \mathrm{H}^1(M, G^* M \otimes \mathrm{End}(E))$$

vanishes.

Atiyah classes II

When M is a regular generalized complex manifold, we have another definition for Atiyah classes. Recall the exact sequence of generalized holomorphic bundles on M:

$$0 \to G^*M \otimes E \to \mathfrak{J}^1 E \to E \to 0.$$

Definition

Let E be a generalized holomorphic vector bundle on a regular generalized complex manifold M. The **Atiyah class** of E is defined to be the first extension class of the above short exact sequence:

$$A(E) \in \mathrm{Ext}^1_M(E, G^*M \otimes E).$$

Theorem

Let E be a generalized holomorphic bundle on a regular generalized complex manifold M. Then E admits a generalized holomorphic connection if and only if $A(E) = 0$, namely, the above short exact sequence splits.

Atiyah classes III

Chen-Stienon-Xu, 2016

For a Lie pair (L, A) and an A-module E, $\Gamma(A) \times \Gamma(E) \to \Gamma(E)$,

(1) Extend the A-module structure to an L-connection ∇ on E, i.e. $\nabla : \Gamma(L) \times \Gamma(E) \to \Gamma(E)$;

(2) The curvature $R^\nabla : \wedge^2 L \to \mathrm{End}(E)$ induces an element $R^\nabla \in \Gamma(A^* \otimes A^\perp \otimes \mathrm{End}(E))$, as $R^\nabla|_{\wedge^2 A} = 0$:

$$R^\nabla(a, \tilde{l}) = \nabla_a \nabla_l - \nabla_l \nabla_a - \nabla_{[a,l]}.$$

Proposition

R^∇ is a 1-cocycle and $[R^\nabla]$ does not depend on the choice of ∇.

We call $[R^\nabla] \in \mathrm{H}^1(A, A^\perp \otimes \mathrm{End}(E))$ **the Atiyah class**.

$$(TM_{\mathbb{C}}, T^{0,1}M), \qquad (TM, F), \qquad (\mathfrak{g}, \mathfrak{h}), \qquad (TM \bowtie M^{\mathfrak{g}}, M^{\mathfrak{g}}).$$

Let E be a generalized holomorphic vector bundle on a regular generalized complex manifold M. Consider the Lie pair

$$(T_{\mathbb{C}}M, \rho(L_-)),$$

A generalized holomorphic vector bundle E is an L_--module and thus a $\rho(L_-)$-module since $\langle \bar{\partial}_L, \ker\rho \rangle = 0$.
Note that

$$\rho(L_-)^{\perp} = L_- \cap T^*_{\mathbb{C}}M = G^*M.$$

So a get the Atiyah class

$$A(E) \in \mathrm{H}^1(\rho(L_-), G^*M \otimes \mathrm{End}(E)).$$

Theorem

This Atiyah class vanishes iff there exists a generalized holomorphic connection on the generalized holomorphic vector bundle E.

Example

Let E be a generalized holomorphic vector bundle over a holomorphic Poisson manifold (M, π).The Atiyah class vanishes iff there exists a holomorphic connection on E such that $D_X = 0$ for any Hamiltonian vector field X on M. In other words, the connection 1-form takes values in the kernel of π.

Example

Let E be a generalized holomorphic vector bundle over a holomorphic Poisson manifold (M, π).The Atiyah class vanishes iff there exists a holomorphic connection on E such that $D_X = 0$ for any Hamiltonian vector field X on M. In other words, the connection 1-form takes values in the kernel of π.

This is different from the definition of the Atiyah class of a holomorphic vector bundle E on a holomorphic Poisson manifold M defined by Chen-Liu-Xiang, 2019, which vanishes iff there is a holomorphic $(T^*M)^{1,0}$-connection on E.

Summary and Outlook

We considered a particular class of generalized holomorphic vector bundles and studied the Atiyah class.

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We may study the Atiyah class of Gualtieri's generalized holomorphic vector bundles. Now we have a Courant algebroid, which is the double of two Dirac structures:

$$(TM \oplus T^*M) \otimes \mathbb{C} = L \oplus L_-,$$

and an L_--module E.

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Can we define the Atiyah class of a Courant algebroid with a Dirac structure?

Problem: the curvature of a Courant connection is not function linear unless $\nabla_{\mathcal{D}f} = 0$, too strong!

Thanks for your attention!

More discussion

Let $\mathcal{C}$ be a Courant algebroid over M and E a vector bundle over M. Then a $\mathcal{C}$-connection on E is a map:

$$\nabla : \Gamma(E) \to \Gamma(\mathcal{C} \otimes E)$$

such that

$$\nabla(fe) = \mathcal{D}f \otimes e + f\nabla e, \qquad \forall f \in C^\infty(M), e \in \Gamma(E),$$

where $\mathcal{D} : C^\infty(M) \to \Gamma(\mathcal{C})$ is given by $\mathcal{D}(f) = \rho^* df$.
The curvature of the Courant algebroid connection ∇ is defined as

$$R^\nabla : \Gamma(\wedge^2\mathcal{C}) \to \Gamma(\mathrm{End}(E)), \qquad R^\nabla(c, c')e = \nabla_c\nabla_{c'}e - \nabla_{c'}\nabla_c e - \nabla_{[c,c']}e.$$

It is $C^\infty(M)$-linear with respect to e since $[\rho(c), \rho(c')] = \rho[c, c']$. The function linear property relative to c fails because the Courant bracket has different Leibniz rule from the Lie bracket.

Lemma

The curvature $R \in \Gamma(\wedge^2\mathcal{C} \otimes \mathrm{End}(E))$ iff $\nabla_{\mathcal{D}f} = 0$ for all $f \in C^\infty(M)$.

Lemma

If the condition $\nabla_{\mathcal{D}f} = 0$ holds for all $f \in C^{\infty}(M)$, then we have the Bianchi identity

$$\nabla(R^{\nabla}) = 0.$$

Let A be a regular Dirac structure of the Courant algebroid $\mathcal{C}$, so A is a Lie algebroid. Let E be an A-module. Let ∇ be an $\mathcal{C}$-connection on E extending the A-connection satisfying that $\nabla_{\mathcal{D}f} = 0$ for all $f \in C^\infty(M)$. If there is such an extension, then we shall get a cohomology class of the Lie algebroid A.
The curvature of ∇ is a bundle map $R^\nabla : \wedge^2\mathcal{C} \to \mathrm{End}E$ defined by

$$R^\nabla(c, c') = \nabla_c\nabla_{c'} - \nabla_{c'}\nabla_c - \nabla_{[c,c']}.$$

As E is an A-module, so R^∇ vanishes when restricting on $\wedge^2 A$. Moreover, by the fact that $\nabla_{\mathcal{D}f} = 0$ and $[\mathcal{D}f, c] = -\frac{1}{2}\mathcal{D}\rho(e)f$, we know $R^\nabla(a, \mathcal{D}f) = 0$. Thus the curvature induces a bundle map

$$R_E^\nabla : A \wedge \frac{\mathcal{C}}{A + \mathcal{D}f} \to \mathrm{End}E$$

given by

$$R_E^\nabla(a, [c]) = R^\nabla(a, c) = \nabla_a\nabla_c - \nabla_c\nabla_a - \nabla_{[a,c]}, \qquad a \in \Gamma(A), c \in \Gamma(\mathcal{C}).$$

Here we identify $(\frac{\mathcal{C}}{A+\mathcal{D}f})^*$ with $\ker\rho_A$, which is a vector bundle since A is regular.

Proposition

(1) The element $R_E^\nabla \in \Gamma(A^* \otimes \ker \rho_A \otimes \mathrm{End}(E))$ is a 1-cocycle in the cohomology of the Lie algebroid A with values in the A-module $\ker \rho_A \otimes \mathrm{End}(E)$;

(2) The cohomology class $\alpha_E \in H^1(A, \ker \rho_A \otimes \mathrm{End}(E))$ does not depend on the choice of $\mathcal{C}$-connection extending the A-module;

(3) The Atiyah class α_E vanishes if and only if there exists an A-compatible $\mathcal{C}$-connection on E.

Let (L, B) be a Lie algebroid pair. Namely, L is a Lie algebroid with a Lie subalgebroid B. Let $\mathcal{C} = L \oplus L^*$ be the associated Courant algebroid with the trivial Lie algebroid structure on L^* and let $A = B \oplus B^{\perp}$. Then A is a regular Dirac structure of $\mathcal{C}$.

Let E be a B-module. It naturally is an A-module with the trivial $B^{\perp}$-action.

Proposition

With the above notations, the Atiyah class of the Courant pair $(\mathcal{C}, A)$ with respect to the A-module E is exactly the Atiyah class of the Lie pair (L, B) with respect to the associated B-module E.