Kontsevich–Duflo type theorem for dg manifolds

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Differential graded Lie algebra

Let $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}^i$ be a ($\mathbb{Z}$-)graded vector space. A **differential graded Lie algebra (dgla)** is a triple $(\mathfrak{g}, d, [-,-])$, where

- $(\mathfrak{g}, d)$ is a cochain complex:

$$\cdots \longrightarrow \mathfrak{g}^i \xrightarrow{d} \mathfrak{g}^{i+1} \longrightarrow \cdots$$

  Denote the degree of x by $|x|$, i.e. $x \in \mathfrak{g}^{|x|}$.

- $(\mathfrak{g}, [-,-])$ is a graded Lie algebra:

$$[-,-] : \mathfrak{g}^i \times \mathfrak{g}^j \to \mathfrak{g}^{i+j}$$
$$[x,y] = -(-1)^{|x||y|}[y,x]$$
$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]]$$

- compatibility condition ($\Rightarrow$ $H(\mathfrak{g}, d)$ is a graded Lie algebra):

$$d[x,y] = [dx,y] + (-1)^{|x|}[x,dy]$$

Example (Polyvector fields)

$M = C^\infty$ manifold

- $\mathfrak{g} = \mathcal{T}^\bullet_{\text{poly}}(M) = \bigoplus_{i\geqslant -1} \mathcal{T}^i_{\text{poly}}(M)$
  $\mathfrak{g}^i = \mathcal{T}^i_{\text{poly}}(M) = \Gamma(\Lambda^{i+1} T_M)$
- Schouten bracket $[-,-] : \mathcal{T}^k_{\text{poly}}(M) \otimes \mathcal{T}^l_{\text{poly}}(M) \to \mathcal{T}^{k+l}_{\text{poly}}(M)$

$$[X, f] = X(f), \quad \forall X \in \mathcal{T}^0_{\text{poly}}(M), f \in \mathcal{T}^{-1}_{\text{poly}}(M) = C^\infty(M)$$

$$[X, Y] = \text{Lie bracket of vector fields}, \quad \forall X, Y \in \mathcal{T}^0_{\text{poly}}(M)$$

$$[\xi, \eta \wedge \zeta] = [\xi, \eta] \wedge \zeta + (-1)^{|\xi||\eta|} \eta \wedge [\xi, \zeta]$$

- d = zero differential, $d = 0 : \mathcal{T}^i_{\text{poly}}(M) \to \mathcal{T}^{i+1}_{\text{poly}}(M)$
- $(\mathcal{T}^\bullet_{\text{poly}}(M), 0, [-,-])$ is a dgla

Example (Polydifferential operators)

$$\mathfrak{g} = \mathcal{D}^{\bullet}_{\text{poly}}(M) = \bigoplus_{i \geqslant -1} \mathcal{D}^{i}_{\text{poly}}(M)$$

- $\mathcal{D}^{i}_{\text{poly}}(M) = \underbrace{\mathcal{D}(M) \otimes_R \cdots \otimes_R \mathcal{D}(M)}_{i+1 \text{ factors}}$
  $R = C^{\infty}(M)$, $\mathcal{D}(M) =$ differential operators on M
- Gerstenhaber bracket
  $$[\![-,-]\!] : \mathcal{D}^{k}_{\text{poly}}(M) \otimes \mathcal{D}^{l}_{\text{poly}}(M) \to \mathcal{D}^{k+l}_{\text{poly}}(M)$$
- Hochschild differential $d_{\mathcal{H}} : \mathcal{D}^{i}_{\text{poly}}(M) \to \mathcal{D}^{i+1}_{\text{poly}}(M)$
- $(\mathcal{D}^{\bullet}_{\text{poly}}(M), d_{\mathcal{H}}, [\![,]\!])$ is a dgla (a sub-dgla of the dgla of Hochschild cochains)

L_∞ morphism

Let $(\mathfrak{g}, d_\mathfrak{g}, [-,-]_\mathfrak{g})$, $(\mathfrak{h}, d_\mathfrak{h}, [-,-]_\mathfrak{h})$ be dglas.

Idea of L_∞ morphism:
Lift Lie algebra morphism $H(\mathfrak{g}, d_\mathfrak{g}) \to H(\mathfrak{h}, d_\mathfrak{h})$ to cochain level.

An L_∞ morphism $\Phi = (\Phi_n)_{n=1}^{\infty} : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is a sequence of linear maps $\Phi_n : \Lambda^n \mathfrak{g} \to \mathfrak{h}$ of degree $1 - n$ such that

- $d_\mathfrak{h}\Phi_1(v_1) = \Phi_1(d_\mathfrak{g} v_1)$
- $\Phi_1([v_1, v_2]_\mathfrak{g}) - [\Phi_1(v_1), \Phi_1(v_2)]_\mathfrak{h} =$
  $d_\mathfrak{h}\Phi_2(v_1 \wedge v_2) - \Phi_2(d_\mathfrak{g} v_1 \wedge v_2) - (-1)^{|v_1|}\Phi_2(v_1 \wedge d_\mathfrak{g} v_2)$
- and higher equations (infinitely many equations)

An L_∞ quasi-isomorphism $\Phi : \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is an L_∞ morphism Φ such that $\Phi_{1*} : H(\mathfrak{g}, d_\mathfrak{g}) \to H(\mathfrak{h}, d_\mathfrak{h})$ is an isomorphism.

Kontsevich formality

Hochschild–Kostant–Rosenberg map (skew-symmetrization)

$$\mathrm{hkr} : \mathcal{T}^{\bullet}_{\mathrm{poly}}(M) \to \mathcal{D}^{\bullet}_{\mathrm{poly}}(M)$$

$$\mathrm{hkr}(X_1 \wedge \cdots \wedge X_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sign}(\sigma)\, X_{\sigma(1)} \otimes \cdots \otimes X_{\sigma(k)}, \quad \forall X_i \in \mathfrak{X}(M)$$

Kontsevich formality theorem

Theorem (Kontsevich, Tamarkin)

- $\exists$ *an* L_∞ *quasi-isomorphism* $\Phi : \mathcal{T}^{\bullet}_{\mathrm{poly}}(M) \rightsquigarrow \mathcal{D}^{\bullet}_{\mathrm{poly}}(M)$ *such that its first Taylor coefficient* $\Phi_1 = \mathrm{hkr}$*;*
- hkr *induces an isomorphism of Gerstenhaber algebras*

$$\mathrm{hkr} : \mathcal{T}^{\bullet}_{\mathrm{poly}}(M) \xrightarrow{\cong} H^{\bullet}(\mathcal{D}^{\bullet}_{\mathrm{poly}}(M), d_{\mathcal{H}}).$$

Application: Duflo type theorems

Theorem (Poincaré–Birkhoff–Witt)

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. The map

$$\mathrm{pbw} : S(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$$

defined by the formula

$$\mathrm{pbw}(X_1 \odot \cdots \odot X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

is an isomorphism of vector spaces.

NOTE:
pbw does NOT preserve the algebra structures.

- In 1970, Duflo modified pbw by Duflo element

$$J := \det\left(\frac{1-e^{-\mathrm{ad}}}{\mathrm{ad}}\right) \in \widehat{S}\mathfrak{g}^\vee,$$

where $\mathrm{ad} \in \mathrm{Hom}(\mathfrak{g}, \mathrm{End}(\mathfrak{g})) \cong \mathfrak{g}^\vee \otimes \mathrm{End}(\mathfrak{g})$, and proved the compatibility with product by techniques of representation theory including Kirillov's orbit method.
- In 1997, Kontsevich proposed a completely different proof by Kontsevich formality theorem.
- Following Kontsevich's idea, Pevzner and Torossian gave a new proof of Duflo. (2004)

Theorem (Duflo)

Let $\mathfrak{g}$ be a Lie algebra. The map

$$\mathrm{pbw} \circ J^{\frac{1}{2}} : S(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$$

is an isomorphism of algebras.

Kontsevich's approach can be applied to different situations including complex manifolds:

Kontsevich–Duflo theorem (Kontsevich, Calaque & Van den Bergh)

$\mathrm{hkr} \circ (\mathrm{Td}_X)^{\frac{1}{2}} : H^\bullet(X, \Lambda^\bullet T_X) \to HH^\bullet(X)$ *is an isomorphism of Gerstenhaber algebras.*

Today:
Unify these theorems by dg manifolds and Atiyah classes.

Differential graded manifolds

A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with support M is a sheaf $\mathcal{R}$ of commutative $\mathbb{Z}$-graded algebras over a smooth manifold M such that $\mathcal{R}(U) \cong C^\infty(U) \otimes S(V^\vee)$ for sufficiently small open subsets U of M and some $\mathbb{Z}$-graded vector space V.

We say $\mathcal{M}$ is finite-dimensional if $\dim M + \dim V < \infty$.

$C^\infty(\mathcal{M}) := \mathcal{R}(\mathcal{M})$

Definition

A dg manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a vector field $Q \in \mathfrak{X}(\mathcal{M}) = \mathrm{Der}(C^\infty(\mathcal{M}))$ of degree $+1$ such that $[Q, Q] = 2\, Q \circ Q = 0$.

Example

If $\mathfrak{g}$ is a Lie algebra, then $(\mathcal{M} = \mathfrak{g}[1], Q = d_{\mathrm{CE}})$ is a dg manifold:

- function algebra: $C^\infty(\mathcal{M}) \cong \Lambda^\bullet \mathfrak{g}^\vee$
- cohomological vector field: $Q = d_{\mathrm{CE}} : \Lambda^\bullet \mathfrak{g}^\vee \to \Lambda^{\bullet+1} \mathfrak{g}^\vee$

Example

If X is a complex manifold, then $(\mathcal{M} = T_X^{0,1}[1], Q = \bar{\partial})$ is a dg manifold:

- function algebra: $C^\infty(T_X^{0,1}[1]) \cong \Omega^{0,\bullet}(X)$
- cohomological vector field: $Q = \bar{\partial}$ (Dolbeault operator)

Why dg manifolds?

- A physical motivation: AKSZ.
- Many interesting examples: Lie algebras, foliations, Lie algebroids, Courant aglebroids, etc.
- Derived Differential Geometry: negatively graded dg manifolds.

Example

Let E be a vector bundle over M and $s \in \Gamma(M, E)$. Then $(\mathcal{M} = E[-1], Q = \iota_s)$ is a dg manifold: derived zero locus of s.

- function algebra:

$$\cdots \xrightarrow{\iota_s} \Gamma(\Lambda^2 E^\vee) \xrightarrow{\iota_s} \Gamma(\Lambda^1 E^\vee) \xrightarrow{\iota_s} C^\infty(M) \longrightarrow 0$$

which is quasi-isomorphic to $C^\infty(Z(s))$ (as cdgas) if s is a regular section.

Theorem (Behrend, L, Xu)

The category of finite-dimensional negatively graded dg manifolds is a category of fibrant objects.

Atiyah class for dg manifold

Atiyah (1957): obstruction to the existence of holomorphic connections on a holomorphic vector bundle

For dg manifold:

- $(\mathcal{M}, Q) =$ dg manifold, $\nabla =$ affine connection on $\mathcal{M}$
- **Atiyah cocycle:** $\mathrm{at}^{\nabla}_{(\mathcal{M},Q)} \in \Gamma(T^{\vee}_{\mathcal{M}} \otimes \operatorname{End} T_{\mathcal{M}})$ given by

$$\mathrm{at}^{\nabla}_{(\mathcal{M},Q)}(X, Y) = L_Q(\nabla_X Y) - \nabla_{(L_Q X)} Y - (-1)^{|X|}\nabla_X(L_Q Y)$$

  where $X, Y \in \mathfrak{X}(\mathcal{M})$. $\quad \mathrm{at}^{\nabla}_{(\mathcal{M},Q)} = L_Q(\nabla)$

- **Atiyah class:**
  $\alpha_{(\mathcal{M},Q)} = [\mathrm{at}^{\nabla}_{(\mathcal{M},Q)}] \in H^1(\Gamma(T^{\vee}_{\mathcal{M}} \otimes \operatorname{End} T_{\mathcal{M}}), L_Q)$

A connection ∇ on $\mathcal{M}$ is called a Q-**invariant connection** if

$$[Q, \nabla_X Y] = \nabla_{[Q,X]} Y + \nabla_X [Q, Y] \qquad \forall X, Y \in \mathfrak{X}(\mathcal{M})$$

Proposition

- *The Atiyah class* $\alpha_{(\mathcal{M},Q)} \in H^1_{\mathrm{CE}}(\Gamma(T^\vee_{\mathcal{M}} \otimes \operatorname{End} T_{\mathcal{M}}), L_Q)$ *does not depend on the choice of connection* ∇.
- $\alpha_{(\mathcal{M},Q)} = 0 \quad \Leftrightarrow \quad \exists$ *Q-invariant connection on* $\mathcal{M}$

Todd class for dg manifold

- Todd cocycle of $(\mathcal{M}, Q)$ associated to ∇ is the L_Q-cocycle

$$\mathrm{td}^{\nabla}_{(\mathcal{M},Q)} = \mathrm{Ber}\left(\frac{\mathrm{at}^{\nabla}_{(\mathcal{M},Q)}}{1-e^{-\mathrm{at}^{\nabla}_{(\mathcal{M},Q)}}}\right) \in \prod_{k\geq 0}\left(\Gamma(\Lambda^k T^{\vee}_{\mathcal{M}})\right)^k$$

- Todd class of a $(\mathcal{M}, Q)$ is the class

$$\mathrm{Td}_{(\mathcal{M},Q)} = \mathrm{Ber}\left(\frac{\alpha_{(\mathcal{M},Q)}}{1-e^{-\alpha_{(\mathcal{M},Q)}}}\right) \in \prod_{k\geq 0} H^k(\Gamma(\Lambda^k T^{\vee}_{\mathcal{M}})^{\bullet}, L_Q).$$

HKR theorem for dg manifolds

Let $(\mathcal{M}, Q)$ be a dg manifold.

Twisted dgla of polyvector fields:

$$\left({}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})\right)_Q := \left({}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M}), 0 + [Q, -], [-, -]\right)$$

Twisted dgla of polydifferential operators:

$$\left({}_{\oplus}\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M})\right)_Q := \left({}_{\oplus}\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M}), d_{\mathcal{H}} + [\![Q, -]\!], [\![-, -]\!]\right)$$

Proposition

The map $\text{hkr} : (\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})^{\bullet}, 0, [Q, -]) \to (\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M})^{\bullet}, d_{\mathcal{H}}, [\![Q, -]\!])$ *is a morphism of double complexes. Moreover, the induced map*

$$\text{hkr} : \mathbb{H}^{\bullet}({}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M}), [Q, -]) \to \mathbb{H}^{\bullet}({}_{\oplus}\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M}), d_{\mathcal{H}} + [\![Q, -]\!])$$

is an isomorphism of vector spaces.

NOTE:
hkr does NOT preserve the algebra structures.

REMEDY:
Modify hkr by Todd class.

Formality theorem for dg manifold

Theorem (L, Stiénon, Xu)

Given a finite-dimensional dg manifold $(\mathcal{M}, Q)$ and an affine torsion-free connection ∇ on $\mathcal{M}$, there exists an L_∞ quasi-isomorphism

$$\Phi : \left({}_{\oplus}\mathcal{T}^{\bullet}_{\text{poly}}(\mathcal{M})\right)_Q \rightsquigarrow \left({}_{\oplus}\mathcal{D}^{\bullet}_{\text{poly}}(\mathcal{M})\right)_Q$$

whose first 'Taylor coefficient' Φ_1 satisfies the following two properties:

- *Φ_1 induces an isomorphism of associate algebras of the cohomologies;*
- *Φ_1 is given by the formula*

$$\Phi_1 = \text{hkr} \circ (\text{td}^{\nabla}_{(\mathcal{M},Q)})^{\frac{1}{2}}.$$

Kontsevich–Duflo theorem for dg manifolds

Corollary

Given a finite-dimensional dg manifold $(\mathcal{M}, Q)$*, the map*

$$\mathrm{hkr} \circ (\mathrm{Td}_{(\mathcal{M},Q)})^{\frac{1}{2}} : \mathbb{H}^\bullet({}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}), [Q, \ ]) \to \mathbb{H}^\bullet({}_\oplus\mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{M}), d_{\mathcal{H}} + [\![Q, \ ]\!])$$

is an isomorphism of Gerstenhaber algebras. The square root $(\mathrm{Td}_{(\mathcal{M},Q)})^{\frac{1}{2}} \in \prod_{k\geq 0} H^k((\Gamma(\Lambda^k T^\vee_{\mathcal{M}}))^\bullet, L_Q)$ *acts on* $\mathbb{H}^\bullet({}_\oplus\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}), [Q, \ ])$ *by contraction.*

Application: Lie algebra

- $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$
- $\mathrm{at}^{\nabla}_{(\mathcal{M},Q)} \in \Gamma(T_{\mathfrak{g}[1]} \otimes \operatorname{End} T_{\mathfrak{g}[1]}) \cong \Lambda^{\bullet}\mathfrak{g}^{\vee} \otimes \mathfrak{g}^{\vee} \otimes \operatorname{End}\mathfrak{g}$:
  $\mathrm{at}^{\nabla}_{(\mathcal{M},Q)} : \mathfrak{g} \ni X \mapsto \mathrm{ad}_X \quad (\nabla = \text{trivial connection})$
- $\mathrm{td}^{\nabla}_{(\mathcal{M},Q)} = \operatorname{Ber}\left(\frac{\mathrm{ad}}{1-e^{-\mathrm{ad}}}\right) = \det\left(\frac{1-e^{-\mathrm{ad}}}{\mathrm{ad}}\right) = J \in \hat{S}\mathfrak{g}^{\vee}$

Theorem

The map $\mathrm{pbw} \circ J^{\frac{1}{2}} : H^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, S(\mathfrak{g})) \to H^{\bullet}_{\mathrm{CE}}(\mathfrak{g}, \mathcal{U}(\mathfrak{g}))$ *is an isomorphism of algebras.*

Application: complex manifold

X = complex manifold, $\quad (\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$

Theorem (Chen, Xiang, Xu)

The following diagram is commutative:

$$\begin{array}{ccc}
\mathbb{H}^\bullet(\oplus \mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M}), [Q, \ ]) & \xrightarrow{\mathrm{hkr}\circ(\mathrm{Td}_{(\mathcal{M},Q)})^{\frac{1}{2}}} & \mathbb{H}^\bullet(\oplus \mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{M}), d_{\mathcal{H}} + [\![Q, \ ]\!]) \\
\cong \downarrow & & \downarrow \cong \\
H^\bullet(X, \Lambda^\bullet T_X) & \xrightarrow{\mathrm{hkr}\circ(\mathrm{Td}_X)^{\frac{1}{2}}} & HH^\bullet(X)
\end{array}$$

Kontsevich–Duflo theorem (Kontsevich, Calaque & Van den Bergh)

$\mathrm{hkr}\circ(\mathrm{Td}_X)^{\frac{1}{2}} : H^\bullet(X, \Lambda^\bullet T_X) \to HH^\bullet(X)$ *is an isomorphism of Gerstenhaber algebras.*

Sketch of proof of Kontsevich–Duflo Thm

We are interested in the isomorphism of Gertenhaber algebras:

$$\mathrm{hkr} \circ (\mathrm{Td}_{(\mathcal{M},Q)})^{\frac{1}{2}} : \mathbb{H}^\bullet\left({}_{\oplus}\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M})_Q\right) \to \mathbb{H}^\bullet\left({}_{\oplus}\mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{M})_Q\right)$$

It factors in the following way

$$\begin{array}{ccc}
\mathbb{H}(\mathfrak{T}^\bullet_{\mathrm{poly}}(\mathcal{F}), [D^\nabla + \tilde{Q}, -]) & \xrightarrow{II} & \mathbb{H}(\mathfrak{D}^\bullet_{\mathrm{poly}}(\mathcal{F}), [\![D^\nabla + m + \tilde{Q}, -]\!]) \\
\uparrow I & & \downarrow III \\
\mathbb{H}\left({}_{\oplus}\mathcal{T}^\bullet_{\mathrm{poly}}(\mathcal{M})_Q\right) & \dashrightarrow & \mathbb{H}\left({}_{\oplus}\mathcal{D}^\bullet_{\mathrm{poly}}(\mathcal{M})_Q\right)
\end{array}$$

where I and III are from *Fedosov contractions*, and II is induced by a twisted (fiberwise, local) Kontsevich formality morphism.
Atiyah classes appear because of the twisting II.
D^∇ is a bridge between local formulas and global formulas.

Classical construction of D^∇

- $\mathcal{M}$ is a graded manifold of dimension n
- $(x_k)_{k\in\{1,\dots,n\}}$ local coordinates on $\mathcal{M}$
- $(y_k)_{k\in\{1,\dots,n\}}$ induced local frame of $T^\vee_{\mathcal{M}}$, regarded as fiberwise linear functions on $T_{\mathcal{M}}$
- Koszul vector field:

$$\delta = \sum_{k=1}^{n} dx_k \frac{\partial}{\partial y_k} \in \Omega^1(\mathcal{M}, \hat{S} T^\vee_{\mathcal{M}} \otimes T_{\mathcal{M}})$$

  regarded as a formal vector field acting on $\Omega(\mathcal{M}, \hat{S} T^\vee_{\mathcal{M}})$.
- Homotopy operator: $h : \Omega^p(\mathcal{M}, S^q T^\vee_{\mathcal{M}}) \to \Omega^{p-1}(\mathcal{M}, S^{q+1} T^\vee_{\mathcal{M}})$

$$h(\omega \otimes f) = \frac{1}{p+q} \sum_{k=1}^{n} (-1)^{|y_k||\omega|} \iota_{\frac{\partial}{\partial x_k}} \omega \otimes y_k \cdot f$$

- Twist δ by d^{∇}, but $(-\delta + d^{\nabla})^2 \neq 0$.
- Consider operators of the form

$$D^{\nabla} = -\delta + d^{\nabla} + X$$

where the correction

$$X \in \Omega^1(\mathcal{M}, \widehat{S}^{\geqslant 2}(T^{\vee}_{\mathcal{M}}) \otimes T_{\mathcal{M}}).$$

- Solve X by the equation $(D^{\nabla})^2 = 0$.

Theorem

Let ∇ be a torsion-free connection. There exists a unique degree one element $X \in \Omega^1(\mathcal{M}, \widehat{S}^{\geqslant 2}(T^{\vee}_{\mathcal{M}}) \otimes T_{\mathcal{M}})$ such that

- $(h \otimes \mathrm{id})(X) = 0$,
- $(D^{\nabla})^2 = 0$.

Our construction of D^∇

Definition

Let $\mathcal{M}$ be a graded manifold. The formal exponential map associated to a connection ∇ on $T_\mathcal{M}$ is the morphism of left $C^\infty(\mathcal{M})$-modules

$$\mathrm{pbw} = \mathrm{pbw}^\nabla : \Gamma(S(T_\mathcal{M})) \to \mathcal{D}(\mathcal{M}),$$

inductively defined by the recursive relations

$$\mathrm{pbw}(f) = f, \qquad \mathrm{pbw}(X) = X,$$

$$\mathrm{pbw}(X_0 \odot \cdots \odot X_n) = \frac{1}{n+1} \sum_{k=0}^{n} \pm \Big\{ X_k \cdot \mathrm{pbw}(X^{\{k\}}) - \mathrm{pbw}\left(\nabla_{X_k} X^{\{k\}}\right) \Big\}$$

$$X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n.$$

- $\mathrm{pbw} = \mathrm{pbw}^{\nabla}$ is the ∞-jet of the geodesic exponential map if $\mathcal{M} = M$.

- $\mathrm{pbw} : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M})$ is an isomorphism of coalgebras over $C^{\infty}(\mathcal{M})$.

Back to construction of D^∇

- Pick a connection ∇ on $T_\mathcal{M}$, and we have a coalgebra isomorphism

$$\mathrm{pbw} = \mathrm{pbw}^\nabla : \Gamma(S(T_\mathcal{M})) \to \mathcal{D}(\mathcal{M}).$$

- Consider the connection $\nabla^\natural$ on $S(T_\mathcal{M})$ defined by

$$\nabla^\natural_X S := \mathrm{pbw}^{-1}\left(X \cdot \mathrm{pbw}(S)\right)$$

for all $X \in \Gamma(T_\mathcal{M})$ and $S \in \Gamma(S(T_\mathcal{M}))$.

- The connection $\nabla^\natural$ induces a connection on the dual bundle $\widehat{S}T^\vee_\mathcal{M}$. By abusing notations, we use the same symbol $\nabla^\natural$ for this induced connection.

The covariant derivative

$$d^{\nabla^{\natural}} : \Omega^p(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}})) \to \Omega^{p+1}(\mathcal{M}, \widehat{S}(T^\vee_{\mathcal{M}}))$$

Theorem (L, Stiénon)

- *The connection* $\nabla^{\natural}$ *is flat. Namely*

$$(d^{\nabla^{\natural}})^2 = 0$$

- *If* ∇ *is a torsion-free connection, then* $d^{\nabla^{\natural}} = -\delta + d^{\nabla} + X$ *with*

$$X \in \Omega^1(\mathcal{M}, \widehat{S}^{\geqslant 2}(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}),$$

  $\mathrm{degree}(X) = +1$, *and* $(h \otimes \mathrm{id})(X) = 0$.

Thus,

$$d^{\nabla^{\natural}} = D^{\nabla}$$

Thank you!