Cup Product and Deformations of A_∞-algebras

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Outline

1. Where A_∞-algebras come from in higher-energy physics?
2. Reminder on A_∞-algebras
3. Braces and $\cup$-product on A_∞-cohomology
4. Families of A_∞-algebras and their inner deformations
5. Examples of inner deformations
6. Cup-product on L_∞-cohomology and the Atiyah class

Physical Motivations:

Higher-Spin Interaction Problem

What is a mathematical structure underlying fundamental interactions?

Massless particles are classified by spin:

- $s = 0$ – scalar field (too simple, no gauge symmetry)
- $s = 1$ – YM fields (geometry of connections in vector bundles)
- $s = 2$ – Einstein's gravity (Riemannian geometry)
- $s > 2$ – ???

There are good relativistic wave equations for free fields of all spins on the Minkowski or (anti-)de Sitter spaces!

Strong Homotopy Algebras are Coming to the Stage

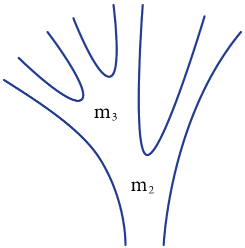
'Quantum Gravity' seems to require higher-spin particles (UV compleation).

Supergravity with $\mathcal{N} > 8$?

String Field Theory:

$$Q\Phi = m_2(\Phi,\Phi) + m_3(\Phi,\Phi,\Phi) + m_4(\Phi,\Phi,\Phi,\Phi) + \cdots ,$$

- Φ – string field (Grassmann odd)
- Q – BRST operator ($Q^2 = 0$)
- m_k – tree level string's amplitudes



A_∞- and L_∞-algebras from String Field Theory

Integrability condition for SFT equations ($Q^2 = 0$) leads to

$$\sum_{k+l=n} \pm m_k(\ldots, m_l(\ldots), \ldots) = 0\,, \quad n = 3, 4, \ldots,$$

i.e., defining conditions of a (minimal) A_∞-algebra constituted by

$$m_k(a_1, a_2, \ldots, a_k)\,, \qquad k = 2, 3, \ldots$$

If all m_k are skew-symmetric, then we get a (minimal) L_∞-algebra.

$$A_\infty \quad \Leftrightarrow \quad \text{(open strings)}\,, \qquad L_\infty \quad \Leftrightarrow \quad \text{(closed strings)}$$

[E. Witten, B. Zwiebach, M. Gaberdiel, H. Kajiura & J. Stasheff, ...]

Higher Spin Gravities

- Φ – differential forms with values in algebra
- $Q = d$ – exterior differential on forms

Field equations [M. Vasiliev, 1988]:

$$d\Phi = m_2(\Phi, \Phi) + m_3(\Phi, \Phi, \Phi) + \cdots$$

$m_2(a, b)$ defines an associative algebra structure, the Higher Spin Algebra.

Problem: given m_2, find all higher interaction vertices m_k's.

Deformation interpretation:

$$m = m_2 + \lambda m_3 + \lambda^2 m_4 + \cdots ,$$

λ being formal deformation parameter (coupling constant).

Graded Vector Spaces and $\circ$-product

- $V = \bigoplus_l V^l$ – $\mathbb{Z}$-graded vector space over k.
- $T(V) = \bigoplus_{n\geq 1} V^{\otimes n}$ – tensor algebra of V.

$$\mathrm{Hom}(T(V), V) = \bigoplus_{l\in\mathbb{Z}} \mathrm{Hom}^l(T(V), V)$$

Non-associative $\circ$-product:

$$(f \circ g)(v_1 \otimes v_2 \otimes \cdots \otimes v_{m+n-1})$$

$$= \sum_{i=0}^{n-1} (-1)^{|g|\sum_{j=1}^{i}|v_j|} f(v_1 \otimes \cdots \otimes v_i \otimes g(v_{i+1} \otimes \cdots \otimes v_{i+m}) \otimes \cdots \otimes v_{m+n-1})$$

$|g|$ is the degree of g as a linear map.

Gerstenhaber Bracket and A_∞-algebras

The Gerstenhaber bracket

$$[f, g] = f \circ g - (-1)^{|f||g|} g \circ f$$

makes $\mathrm{Hom}(T(V), V)$ into the graded Lie superalgebra $\mathcal{L} = \bigoplus_n \mathcal{L}^n$.

- Super skew-symmetry: $[f, g] = -(-1)^{|f||g|}[g, f]$,
- The super Jicobi identity:

$$[[f, g], h] = [f, [g, h]] - (-1)^{|f||g|}[g, [f, h]] \,.$$

An A_∞-structure on V is given by a MC element $m \in \mathcal{L}^1$, i.e.

$$[m, m] = 2m \circ m = 0 \,.$$

(V, m) is called A_∞-algebra [Jim Stasheff, 1963].

Special cases of A_∞-algebras

Stasheff's identities for component maps:

$$m = m_1 + m_2 + m_3 + \cdots\,, \qquad m_n \in \mathrm{Hom}^1(T^n(V), V)\,,$$

$$m\circ m = 0 \quad \Leftrightarrow \quad \sum_{k+l=n-1} m_k \circ m_l = 0\,, \quad n = 1, 2, \ldots$$

- $m = m_1 \Rightarrow (V, d)$ is a complex with differential $d = m_1$; $d^2 = 0$
- $m = m_2 \Rightarrow (V, \cdot\,)$ is an associative algebra with

$$u\cdot v = (-1)^{|u|-1} m_2(u,v)$$

- $m = m_1 + m_2 \Rightarrow (V, d, \cdot\,)$ is a DGA algebra,

$$d(u\cdot v) = du\cdot v + (-1)^{|u|-1} u\cdot dv$$

- $m = m_2 + m_3 + \cdots$ is a minimal A_∞-structure; $m_2\circ m_2 = 0$.

Minimal Deformations of Associative Algebras

The ideal $\bar{\mathcal{L}} = \prod_{n\geq 2} \mathrm{Hom}^0(T^n(V), V) \subset \mathcal{L}$ generates the subgroup of internal automorphisms $G \subset \mathrm{Aut}(\mathcal{L})$:

$$f \mapsto f' = e^{[h,\,\cdot\,]} f = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[h,[h,\dots[h}_{n}, f]\dots]\,, \qquad \forall h \in \bar{\mathcal{L}}\,, \quad \forall f \in \mathcal{L}$$

Two minimal A_∞-structures m and m' are equivalent ($m \sim m'$), if

$$m' = e^{[h,\,\cdot\,]} m \quad \text{for some} \quad h \in \bar{\mathcal{L}}\,.$$

$$\mathcal{M}_V = (\text{Minimal } A_\infty\text{-structures on } V)/G$$

Clearly, $m \sim m' \Rightarrow m_2 = m_2'$.

$\mathcal{M}_V$ may be regarded as the space of all nontrivial deformations of associative algebras $A = (V, m_2)$ in the category of minimal A_∞-algebras.

Braces

Let $W = \mathrm{Hom}(T(V), V)$ and $A_0, A_1, \ldots, A_m \in W$.

Following [Kadeishvili, 1988], define *m*-brace $\{\ \ \}: W^{m+1} \to W$ by

$$A_0\{A_1, \ldots, A_m\}(v_1, \ldots, v_n) = \sum_{Av\text{-shuffles}} \pm A_0(v_1, \ldots, v_{k_1}, A_1(v_{k_1+1}, \ldots) \ldots, v_{k_m}, A_m(v_{k_m+1}, \ldots) \ldots, v_n)$$

By definition, $A\{\varnothing\} = A$. Clearly, $A_0\{A_1\} = A_0 \circ A_1$.

Higher pre-Jacobi identities [Gerstenhaber & Voronov, 1995]:

$$A\{A_1, \ldots, A_m\}\{B_1, \ldots, B_n\} = \sum_{AB\text{-shuffles}} \pm A\{B_1, \ldots, B_{k_1}, A_1\{B_{k_1+1}, \ldots\}, \ldots, B_{k_m}, A_m\{B_{k_m+1}, \ldots\}, \ldots, B_n\}$$

Derived A_∞-structure and A_∞-cohomology

Any A_∞-structure m on V can be lifted to an A_∞-structure M on W:

$$M_1(A) = m \circ A - (-1)^{|A|} A \circ m \,,$$

$$M_k(A_1, \dots, A_k) = m\{A_1, \dots, A_k\} \,, \quad k > 1 \,; \qquad M \circ M = 0$$

[E. Getzler, 1993]. In particular, $M_1 \circ M_1 = M_1 M_1 = 0$.

Let $H^\bullet(W)$ denote the cohomology of the complex $M_1 : W^p \to W^{p+1}$.

It follows from $[M_1, M_2] = 0$ that $M_2 : W \otimes W \to W$ induces a product in A_∞-cohomology:

$$\cup : H^n(W) \otimes H^m(W) \to H^{n+m+1}(W) \,,$$

$$a \cup b = (-1)^{|A|-1} M_2(A, B) \,, \qquad a = [A] \,, \quad b = [B] \,.$$

For $A = (V, m_2)$ this yields the usual groups of Hochschild cohomology $HH^{\bullet+1}(A, A)$ endowed with the Gerstenhaber ∪-product.

Further Properties of Cup-product

Theorem. *The cup-product and the Gerstenhaber bracket endow the space $H^\bullet(W)$ of A_∞-cohomology with a structure of the Gerstenhaber algebra.*

- Associativity: $(a \cup b) \cup c = a \cup (b \cup c)$.
- Graded commutativity: $a \cup b = (-1)^{(|a|-1)(|b|-1)} b \cup a$.
- Poisson relation: $[a, b \cup c] = [a, b] \cup c + (-1)^{|a|(|b|+1)} b \cup [a, c]$.

[Sh. & Skvortsov, 2019]

The formal deformations of A_∞-structures are controlled by the groups $H^1(W)$ and $H^2(W)$.

[Penkava & Schwarz, 1995; Fialowski & Penkava, 2002]

Inner A_∞-cohomology of Families

Let (V, m_t) be an n-parameter family of A_∞-algebras, i.e.,

$$m \in W[[t_1, \dots, t_n]]\,, \qquad m \circ m = 0\,, \qquad |m| = 1\,,$$

where $|t_i| \in 2\mathbb{Z}$. Denote $m_{(i)} = \partial m / \partial t_i$, then

$$\frac{\partial}{\partial t^i}(m \circ m) = [m_{(i)}, m] = M_1(m_{(i)}) = 0 \quad \Rightarrow \quad [m_{(i)}] \in H^\bullet(W).$$

The cocycles $m_{(i)}$ generate a commutative algebra D_m w.r.t. the cup-product:

$$D_m \ni \Delta = \sum_{l=0}^{L} c^{i_1 \dots i_l} m_{(i_1)} \cup m_{(i_2)} \cup \dots \cup m_{(i_l)}\,, \qquad c^{i_1 \dots i_l} \in k[[t_1, \dots, t_n]].$$

$[\Delta] \in H^\bullet(W)$ is an inner cohomology class of the family (V, m_t).

Inner Deformations of Families

Proposition. *Any inner cocycle $\Delta[m_t]$ associated to an n-parameter family $m_t \in W[[t_1, \dots, t_n]]$ of A_∞-structures gives rise to an $(n+1)$-parameter family of A_∞-structures $\tilde{m}_t \in W[[t_0, t_1, \dots, t_n]]$. The latter is defined as a unique formal solution to*

$$\tilde{m}_{(0)} = \Delta[\tilde{m}]\,, \qquad \tilde{m}|_{t_0=0} = m_t\,.$$

Here t_0 is a new formal parameter with $|t_0| = 1 - |\Delta|$.

The A_∞-structure $\tilde{m}_t$ is called an inner deformation of the family m_t.

Geometrically, $\Delta[m]$ defines a flow in the space $W[[t_1, \dots, t_n]]$ which is tangent to the quadratic surface Σ of A_∞-structures:

$$\Sigma : m \circ m = 0\,, \qquad L_\Delta(m \circ m) = [\Delta[m], m] = 0\,, \qquad \forall m \in \Sigma\,.$$

Example: Minimal Deformations of DGA's

Let $A_t = (V, \partial, \mu)$ be a 1-parameter family of DGA's, i.e.,

$$V = \oplus_l V^l\,, \qquad \partial : V^l \to V^{l-1}\,, \qquad \mu : V^n \otimes V^m \to V^{n+m}\,.$$

Both the product and the differential may depend on t with $|t| = 0$.

Define the 2-parameter family of DGA's $A_t \otimes k[[u]]$ with the differential

$$d = u\partial\,, \qquad |u| = 2 \quad \Rightarrow \quad |d| = 1\,.$$

Then $m = d + \mu$ gives rise to a sequence of inner cocycles of degree 1

$$\Delta_n[m] = m_{(t)} \cup \underbrace{m_{(u)} \cup m_{(u)} \cup \cdots \cup m_{(u)}}_{n}, \quad n = 1, 2, \dots, \quad (|\cup| = -1)$$

generating the flows

$$\tilde{m}_{(s)} = \Delta_n[\tilde{m}]\,, \qquad \tilde{m}|_{s=0} = m\,.$$

Minimal Deformations of DGA's

The parameter u plays an auxiliary role. Let

$$\bar{m} = \tilde{m}|_{u=0} = m + s\varphi + O(s^2)\,,$$

where

$$\Phi(v_1, \ldots, v_{n+2}) = \mu'(v_1, v_2) \cdot \partial v_3 \cdot \partial v_4 \cdots \partial v_{n+2}$$

and the prime stands for the derivative w.r.t. t.

Φ is a Hochschild cocycle defining an element of $HH^{n+2}(A_t, A_t)$.

Setting $s = 1$, we get a minimal deformation of the graded associative algebra A_t.

If ∂ is independent of t, then ∂ differentiates $\bar{m}$, i.e., $[\partial, \bar{m}] = 0$.

Application: Higher Spin Gravity in D=4

A 2-parameter family of Higher Spin Algebras is $\mathfrak{A} = S \otimes S$, where S is a symplectic reflection algebra [P. Etingof & V. Ginzburg, 2001]:

$$[q,p] = 1 + t\kappa\,, \qquad \{\kappa, q\} = \{\kappa, p\} = 0\,, \qquad \kappa^2 = 1\,, \qquad t \in \mathbb{R}$$

[E. Wigner, 1950]. For $t = 0$, all quadratic monomials in q's and p's generate $sp(2,\mathbb{R}) \simeq so(3,2)$, the Lie algebra of isometry group of AdS_4.

Regarding $\mathfrak{A}$ as a bimodule over itself, define the family of DGA's:

$$A = A_{-1} \oplus A_0, \quad A_{-1} = \mathfrak{A} = \mathcal{A}_0, \quad \partial = id : A_0 \to A_{-1}, \quad \partial^2 = 0.$$

A minimal deformation of the family A gives rise to a $4D$ HS gravity.

Deformation of Maurer–Cartan Space

Given an A_∞-algebra (V, m), the Maurer–Cartan equation reads

$$m(a) := \sum_{n=1}^{\infty} m_n(a, \dots, a) = 0\,, \qquad |a| = 0\,.$$

A solution $a \subset V^0$ is called a Maurer–Cartan element of (V, m).

Proposition. *If $\tilde{m}$ is an inner deformation of a family of A_∞-structures m, then each MC element for m can be deformed to that for $\tilde{m}$.*

Proof: Let $\tilde{m} \in W[[t_0, t_1, \dots, t_l]]$ be a solution to

$$\tilde{m}_{(0)} = \Delta[\tilde{m}_{(i_1)}, \tilde{m}_{(i_2)}, \dots, \tilde{m}_{(i_l)}]\,, \qquad \Delta[\tilde{m}] = \tilde{m}_{(i_1)} \cup \tilde{m}_{(i_2)} \cup \dots \cup \tilde{m}_{(i_l)}\,.$$

Then, any solution to

$$D_0 a = (-1)^l \Delta(D_{i_1}, \tilde{m}_{(i_s)}, \dots, \tilde{m}_{(i_l)})(a)\,, \qquad D_i a := \partial a / \partial t^i\,.$$

satisfies $\tilde{m}(a) = 0$ provided that $a_0 = a|_{t_0=0}$ obeys $m(a_0) = 0$.

Cup-product on L_∞-cohomology

L_∞-structure $\Leftrightarrow$ homological vector field Q on a $\mathbb{Z}$-graded manifold M:

$$|Q| = 1\,, \qquad Q^2 = \frac{1}{2}[Q, Q] = 0\,.$$

The Lie derivative L_Q makes the tensor algebra $T^{\bullet,\bullet}(M)$ into a complex. For any symmetric connection ∇ on M with curvature $R \in T^{3,1}(M)$, the tensor $B_1 \in T^{2,1}(M)$ defined by

$$B_1(X, Y) = \nabla_X \nabla_Y Q - \nabla_{\nabla_X Y} Q - R_{XQ} Y$$

is Q-invariant. (Actually, $B_1 = L_Q \nabla$, and hence $L_Q B_1 = 0$.)

[S. Lyakhovich, E. Mosmann & A. Sh, 2004]

The space $H^{0,1}(M, Q)$, generated by Q-invariant vector fields on M, is endowed with the grad. commutative product $X \cup Y = B_1(X, Y)$, which is compatible with the commutator $[X, Y]$ of Q-invariant vector fields.

The Atiyah class of tangent bundles

The Dolbeualt model of the Atiyah class:

- M – a complex analytic manifold with holomorphic coordinates z^A
- ∇ – a C^∞ affine connection on M of type $(1,0)$
- $\bar\partial$ – the Dolbeault operator defining holomorphic structure on M

$$\mathrm{At}_M = [\bar\partial, \nabla] \in \Omega^{1,1}\otimes \mathrm{End}(TM)\,, \qquad \bar\partial \mathrm{At}_M = 0\,.$$

[M. Kapranov, 1997; R. Mehta, M. Stiénon, P. Xu, 2015]

The complex supermanifold $\mathcal{M} = \Pi TM$ with holomorphic coordinates $(z^A, \theta^A = dz^A)$ is endowed with the canonical homological vector field

$$Q = \bar\theta^A \frac{\partial}{\partial \bar z^A} \qquad \Leftrightarrow \qquad \bar\partial$$

∇ lifts canonically to an affine connection $\hat\nabla$ on $\mathcal{M}$ and we define

$$B_1 = L_Q \hat\nabla \in T^{2,1}(\mathcal{M})\,, \qquad B_1 \sim \mathrm{At}_M\,.$$