Atiyah class of a dg vector bundle relative to a dg Lie algebroid

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1. Infinite jet of exponential map

2. Generalization to graded manifolds

3. If $\mathcal{M}$ is a dg mfd, then $\mathfrak{X}(\mathcal{M})$ is an L_∞ algebra.

4. Atiyah class of a dg vector bundle relative to a dg Lie algebroid

1 Infinite jet of exponential map

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4 Atiyah class of a dg vector bundle relative to a dg Lie algebroid

Exponential maps arise naturally in relation with linearization problems:

1. Lie theory
2. smooth manifolds

PBW isomorphism in Lie theory

- $\mathfrak{g}$, a finite dimensional Lie algebra
- $\exp : \mathfrak{g} \to G$
- $\exp$ is a local diffeomorphism from nbhd of 0 to nbhd of 1
- induced map on distributions $(\exp)_* : \mathcal{D}'(0) \xrightarrow{\cong} \mathcal{D}'(1)$
- canonical identifications: $\mathcal{D}'(0) \cong S\mathfrak{g}$ and $\mathcal{D}'(1) \cong U\mathfrak{g}$
- $\mathrm{pbw} : S\mathfrak{g} \xrightarrow{\cong} U\mathfrak{g}$, Poincaré–Birkhoff–Witt isomorphism

PBW isomorphism in Lie theory

- $\mathfrak{g}$, a finite dimensional Lie algebra
- Poincaré–Birkhoff–Witt map:

$$S\mathfrak{g} \xrightarrow{\text{pbw}} U\mathfrak{g}$$

is the symmetrization map

$$X_1 \odot \cdots \odot X_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

- Fact: $S\mathfrak{g} \xrightarrow{\text{pbw}} U\mathfrak{g}$ is an isomorphism of coalgebras.

Geodesic exponential map and PBW isomorphism

- torsionfree connection ∇ on smooth manifold M
- $\exp^{\nabla} : T_M \to M \times M$ (bundle map)
  defined by $\exp^{\nabla}(X_m) = (m, \gamma(1))$ where γ is the smooth path in M satisfying $\dot{\gamma}(0) = X_m$ and $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$
- $\Gamma(S(T_M))$ seen as space of differential operators on T_M, all derivatives in the direction of the fibers, evaluated along the zero section of T_M
- $\mathcal{D}(M)$ seen as space of differential operators on $M \times M$, all derivatives in the direction of the fibers, evaluated along the diagonal section $M \to M \times M$
- map induced by $\exp^{\nabla}$ on fiberwise differential operators:
  $\mathrm{pbw}^{\nabla} := \exp^{\nabla}_{*} : \Gamma(S(T_M)) \xrightarrow{\cong} \mathcal{D}(M)$ is an isomorphism of left modules over $C^{\infty}(M)$ called Poincaré–Birkhoff–Witt isomorphism

pbw as infinite jet of $\exp$

The Taylor series of the composition

$$T_m M \xrightarrow{\exp^\nabla} \{m\} \times M \xrightarrow{f} \mathbb{R}$$

at the point $0_m \in T_m M$ is

$$\sum_{J \in \mathbb{N}_0^n} \tfrac{1}{J!} \big(\mathrm{pbw}^\nabla(\partial_x^J) f \big)(m) \cdot y^J \quad \in \hat{S}(T_m^\vee M),$$

where

- $(x_i)_{i \in \{1,\dots,n\}}$ are local coordinates on M
- $(y_j)_{j \in \{1,\dots,n\}}$ induced local frame of $T_M^\vee$ regarded as fiberwise linear functions on T_M

Hence pbw^∇ is the fiberwise infinite jet of the bundle map $\exp^\nabla : T_M \to M \times M$ along the zero section of $T_M \to M$.

Algebraic characterization of pbw^∇

Theorem (Laurent-Gengoux, S, Xu, 2014): The map pbw^∇ is the isomorphism of left $C^\infty(M)$-modules $\Gamma(ST_M) \to \mathcal{D}(M)$ satisfying

$$\mathrm{pbw}^\nabla(f) = f, \quad \forall f \in C^\infty(M);$$
$$\mathrm{pbw}^\nabla(X) = X, \quad \forall X \in \mathfrak{X}(M);$$
$$\mathrm{pbw}^\nabla(X^{n+1}) = X \cdot \mathrm{pbw}^\nabla(X^n) - \mathrm{pbw}^\nabla(\nabla_X X^n), \quad \forall n \in \mathbb{N}.$$

Therefore, for all $n \in \mathbb{N}$ and $X_0, \dots, X_n \in \mathfrak{X}(M)$,

$$\mathrm{pbw}^\nabla(X_0 \odot \cdots \odot X_n) = \tfrac{1}{n+1} \sum_{k=0}^{n} \Big\{ X_k \cdot \mathrm{pbw}^\nabla(X^{\{k\}}) - \mathrm{pbw}^\nabla\left(\nabla_{X_k}(X^{\{k\}})\right) \Big\}$$

where $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

Both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ are left coalgebras over $R := C^\infty(M)$.

The comultiplication $\Delta : \mathcal{D}(M) \to \mathcal{D}(M) \otimes_R \mathcal{D}(M)$ is defined by

$$\Delta(D)(f, g) = D(f \cdot g), \quad \forall f, g \in R.$$

Comultiplication in both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ by deconcatenation:

$$\begin{aligned}\Delta(X_1 \cdots X_n) = 1 \otimes (X_1 \cdots X_n) \\ + \sum_{\substack{p+q=n \\ p,q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} (X_{\sigma(1)} \cdots X_{\sigma(p)}) \otimes (X_{\sigma(p+1)} \cdots X_{\sigma(n)}) \\ + (X_1 \cdots X_n) \otimes 1\end{aligned}$$

for all $X_1, \ldots, X_n \in \mathfrak{X}(\mathcal{M})$.

Proposition: $\mathrm{pbw}^\nabla : \Gamma(S(T_M)) \to \mathcal{D}(M)$ is an isomorphism of coalgebras over $C^\infty(M)$.

- $\left(\mathrm{pbw}^{\nabla}\right)^{-1} : \mathcal{D}(M) \to \Gamma(S(T_M))$ takes a differential operator to its *complete symbol*
- both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ are bi-algebroids
- but pbw^{∇} does not respect the algebra structures

What about replacing the smooth manifold M by a differential graded manifold $\mathcal{M}$?

Differential graded manifolds

Definition: A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with base manifold M is a sheaf $\mathcal{R}$ (over M) of $\mathbb{Z}$-graded commutative algebras such that $\mathcal{R}(U) \cong C^\infty(U) \otimes S(V^\vee)$ for sufficiently small open subsets U of M and some $\mathbb{Z}$-graded vector space V. Here $S(V^\vee)$ denotes the graded algebra of polynomials on V.

$C^\infty(\mathcal{M}) := \mathcal{R}(\mathcal{M})$

Theorem (Batchelor): There exists a (noncanonical) $\mathbb{Z}$-graded vector bundle $E \to M$ such that $\mathcal{R}(U) = \Gamma(U; S(E^\vee))$.

Definition: A dg manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a vector field $Q \in \mathfrak{X}(\mathcal{M})$ of degree $+1$ such that $[Q, Q] = 2\ Q \circ Q = 0$.

Example: If $\mathfrak{g}$ is a Lie algebra, then $\mathcal{M} = \mathfrak{g}[1]$ is a dg manifold.

- Its algebra of functions: $C^\infty(\mathfrak{g}[1]) \cong \Lambda^\bullet(\mathfrak{g}^\vee)$.
- Its homological vector field: $Q = d_{\mathrm{CE}}$.

Example: If M is a smooth manifold, then $\mathcal{M} = T_M[1]$ is a dg manifold.

- Its algebra of functions: $C^\infty(T_M[1]) \cong \Omega^\bullet(M)$.
- Its homological vector field: $Q = d_{\mathrm{dR}}$.

Example: If X is a complex manifold, then $\mathcal{M} = T_X^{0,1}[1]$ is a dg manifold.

- Its algebra of functions: $C^\infty(T_X^{0,1}[1]) \cong \Omega^{0,\bullet}(X)$.
- Its homological vector field: $Q = \bar{\partial}$.

Example (Vaĭntrob): For a vector bundle $A \to M$, $(A[1], Q)$ is a dg-manifold $\iff$ A is a Lie algebroid, $d_{\mathrm{CE}} = Q$.

Example: A curved L_∞ algebra structure on a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a coderivation Q of degree $+1$ of the symmetric tensor **co**algebra $S(\mathfrak{g}[1])$ satisfying $Q \circ Q = 0$.

$$
\begin{array}{ccccccc}
 & & S(\mathfrak{g}[1]) & \xrightarrow{\quad Q \quad} & S(\mathfrak{g}[1])[1] & & \\
 & & \cong \updownarrow & & \cong \updownarrow & & \\
(\Lambda^k \mathfrak{g})[k] & \hookrightarrow & \bigoplus_{k \in \mathbb{Z}} (\Lambda^k \mathfrak{g})[k] & \overset{Q}{\dashrightarrow} & \bigoplus_{k \in \mathbb{Z}} (\Lambda^k \mathfrak{g})[k+1] & \twoheadrightarrow & \mathfrak{g}[2]
\end{array}
$$

$$(\Lambda^k \mathfrak{g})[k] \xrightarrow{\lambda_k} \mathfrak{g}[2]$$

The maps $\lambda_k : \Lambda^k \mathfrak{g} \to \mathfrak{g}[2-k]$ $(k = 0, 1, 2, \dots)$ satisfy some axioms.

Ignoring technicalities, dualizing the coalgebra $S(\mathfrak{g}[1])$ and the coderivation Q, we obtain the algebra of functions on the graded mfd $\mathfrak{g}[1]$ and a homological v. f. on it.

Example: Given a regular foliation F, the tangent bundle of F is a subbundle of T_M, denoted T_F, whose sections are closed under the Lie bracket of vector fields, i.e. an integrable distribution of the manifold M.
Then $(\mathcal{M} = T_F[1], Q = d_{\mathrm{dR}})$ is a dg manifold:

- its algebra of functions: $C^\infty(T_F[1]) = \Omega_F^\bullet$, the space of leafwise differential forms;
- its homological vector field: the de Rham differential $Q = d_{\mathrm{dR}}$.

Example: Let s be a smooth section of a vector bundle $E \to M$.
Then $(\mathcal{M} = E[-1], Q = i_s)$ is a dg manifold: the derived intersection of s with the zero section.
Its algebra of functions: $C^\infty(E[-1]) \cong \bigoplus_{k=0}^\infty \Gamma(\Lambda^k(E^\vee))[k]$.

For instance, if $f \in C^\infty(M)$, then $(T_M^\vee[-1], i_{df})$ is a dg manifold called derived critical locus of f.

Formal exponential map

Definition: A connection on a graded mfd $\mathcal{M}$ is a $\mathbb{k}$-linear map

$$\nabla : \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

of degree 0 satisfying

$$\nabla_{fX} Y = f \nabla_X Y,$$
$$\nabla_X (fY) = X(f) Y + (-1)^{|X||f|} f \nabla_X Y,$$

for all $f \in C^\infty(\mathcal{M})$ and all homogeneous $X, Y \in \mathfrak{X}(\mathcal{M})$.

- Geodesics? Not so easy.
- Shortcut: The algebraic relations satisfied by pbw serve as an alternative definition.
- The isomorphism pbw is a sort of formal exponential map defined inductively.

Definition: Let $\mathcal{M}$ be a graded manifold. The formal exponential map associated to a connection ∇ on $T_{\mathcal{M}}$ is the morphism of left $C^\infty(\mathcal{M})$-modules

$$\mathrm{pbw}^\nabla : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M}),$$

inductively defined by the relations

$$\mathrm{pbw}^\nabla(f) = f \qquad \forall f \in C^\infty(\mathcal{M}),$$
$$\mathrm{pbw}^\nabla(X) = X \qquad \forall X \in \Gamma(T_{\mathcal{M}}),$$

and, for all $n \in \mathbb{N}$ and homogeneous $X_0, \dots, X_n \in \Gamma(T_{\mathcal{M}})$,

$$\mathrm{pbw}^\nabla(X_0 \odot \cdots \odot X_n) = \tfrac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \left\{ X_k \cdot \mathrm{pbw}^\nabla(X^{\{k\}}) - \mathrm{pbw}^\nabla\left(\nabla_{X_k} X^{\{k\}}\right) \right\}.$$

- $\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$
- $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$

Proposition (Liao, S, 2015): The formal exponential map

$$\mathrm{pbw}^{\nabla} : \Gamma(S^{\leqslant k}(T_{\mathcal{M}})) \to \mathcal{D}^{\leqslant k}(\mathcal{M})$$

is a well defined isomorphism of filtered coalgebras over $C^{\infty}(\mathcal{M})$.

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If $\mathcal{M}$ is a dg mfd, then $\mathfrak{X}(\mathcal{M})$ is an L_∞ algebra.

1. Given a dg manifold $(\mathcal{M}, Q)$.

2. $\mathcal{L}_Q$ is a coderivation of $\mathcal{D}(\mathcal{M})$ of degree $+1$:

$$\mathcal{L}_Q(X_1 \cdots X_n) = \sum_{k=1}^{n} (-1)^{|X_1|+\cdots+|X_{k-1}|} X_1 \cdots X_{k-1}[Q, X_k] X_{k+1} \cdots X_n.$$

3. Choose torsionfree connection ∇ on $\mathcal{M}$.
   Get isomorphism of coalgebras $\mathrm{pbw} : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M})$.

4. $\delta^\nabla := \left(\mathrm{pbw}^\nabla\right)^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw}^\nabla$
   δ^∇ is a coderivation of $\Gamma(S(T_{\mathcal{M}}))$ of degree $+1$

5. Dualizing δ^∇, we obtain $D^\nabla : \Gamma(\widehat{S}(T^\vee_{\mathcal{M}})) \to \Gamma(\widehat{S}(T^\vee_{\mathcal{M}}))$.
   D^∇ is a derivation of $\Gamma(\widehat{S}(T^\vee_{\mathcal{M}}))$ of degree $+1$

Given torsionfree connection ∇ on dg mfd $(\mathcal{M}, Q)$.

Theorem (Mehta, S, Xu, 2015):

- The operator D^∇ is a derivation of degree $+1$ of the graded algebra $\Gamma(\widehat{S}(T^\vee_{\mathcal{M}}))$ satisfying $(D^\nabla)^2 = 0$.
- There exist $R_k \in \mathrm{Hom}\left(S^k T_{\mathcal{M}}, T_{\mathcal{M}}[1]\right)$ for $k = 2, 3, 4, \dots$ such that $D^\nabla = \mathcal{L}_Q + \sum_{k=2}^\infty R_k^\top$.
- $R_2 \in \mathrm{Hom}\left(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}[1]\right)$ is given by $R_2(X, Y) = \mathcal{L}_Q(\nabla_X Y) - \nabla_{\mathcal{L}_Q X} Y - (-1)^{|X|} \nabla_X(\mathcal{L}_Q Y)$

Corollary (Mehta, S, Xu, 2015): The sequence of operations $(R_k)_{k=1,2,3,\dots}$ where

- $R_1 := \mathcal{L}_Q : \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$
- $R_2 := \mathcal{L}_Q \nabla \in \mathrm{Hom}\left(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}[1]\right)$
- $R_k \in \mathrm{Hom}\left(S^k T_{\mathcal{M}}, T_{\mathcal{M}}[1]\right)$ for $k \geqslant 3$

turn the space of vector fields $\mathfrak{X}(\mathcal{M})$ into an $L_\infty[1]$ algebra.

- The above result is analogous to a theorem of Kapranov about the Atiyah class of Kähler manifolds.

- A theorem of Kapranov states that for a complex manifold X, the complex of sheaves $T_X[-1]$ is a Lie algebra object in the derived category $D(X)$ of coherent sheaves on X with the Atiyah class α_{T_X} playing the role of the Lie bracket.

- If the complex manifold X is Kähler, Kapranov proved an even stronger result by describing explicitly an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$.

If X is a Kähler manifold, the Levi-Civita connection ∇^{LC} induces a $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ as follows. First, extend the Levi-Civita connection $\mathbb{C}$-linearly to a $T_X^{\mathbb{C}}$-connection ∇ on $T_X^{\mathbb{C}}$. Since X is Kähler, the almost complex structure J on X is parallel and ∇ restricts to a $T_X^{\mathbb{C}}$-connection on $T_X^{1,0}$. It is easy to check that the induced $T_X^{0,1}$-connection on $T_X^{1,0}$ is the canonical flat connection $\nabla^{\overline{\partial}}$ encoding the holomorphic vector bundle structure on T_X while the induced $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ is flat and torsion-free. Thus $\nabla = \nabla^{\overline{\partial}} + \nabla^{1,0}$.

The element $R^\nabla \in \Omega^{0,1}\big((T_X^{1,0})^\vee \otimes \mathrm{End}(T_X^{1,0})\big)$ defined by the equation

$$R^\nabla(Z,V)W = \nabla_Z\nabla_V W - \nabla_V\nabla_Z W - \nabla_{[Z,V]}W,$$

for all $Z \in \Gamma(T_X^{0,1})$ and $V, W \in \Gamma(T_X^{1,0})$, is a Dolbeault 1-cocycle representative of the Atiyah class of the holomorphic tangent bundle T_X.

Since $\nabla^{1,0}$ is torsion-free, R^∇ belongs to $\Omega^{0,1}\big(S^2(T_X^{1,0})^\vee \otimes T_X^{1,0}\big)$.

Theorem (Kapranov): Given a Kähler manifold X, the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ admits a structure of $L_\infty[1]$ algebra whose unary bracket λ_1 is the Dolbeault operator $\overline{\partial}:\Omega^{0,j}(T_X^{1,0})\to\Omega^{0,j+1}(T_X^{1,0})$ and whose k-th multibracket λ_k for $k\geqslant 2$ is the composition of the wedge product

$$\Omega^{0,j_1}(T_X^{1,0})\otimes\cdots\otimes\Omega^{0,j_n}(T_X^{1,0})\to\Omega^{0,j_1+\cdots+j_k}\big((T_X^{1,0})^{\otimes k}\big)$$

with the map

$$\Omega^{0,j_1+\cdots+j_k}\big((T_X^{1,0})^{\otimes k}\big)\to\Omega^{0,j_1+\cdots+j_n+1}(T_X^{1,0})$$

induced by

$$R_k\in\Omega^{0,1}\big(S^k(T_X^{1,0})^\vee\otimes T_X^{1,0}\big)\subset\Omega^{0,1}\big(\operatorname{Hom}\big((T_X^{1,0})^{\otimes k},T_X^{1,0}\big)\big)$$

with $R_2=R^\nabla$ and $R_{k+1}=d^{\nabla^{1,0}}R_k$ for $k\geqslant 2$.

Theorem (Laurent, S, Xu): Given a complex manifold X, each torsion-free $T_X^{1,0}$-connection $\nabla^{1,0}$ on $T_X^{1,0}$ determines an $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ such that

- the unary bracket λ_1 is the Dolbeault operator $\overline{\partial} : \Omega^{0,j}(T_X^{1,0}) \to \Omega^{0,j+1}(T_X^{1,0})$;
- the binary bracket λ_2 is the map

$$\lambda_2 : \Omega^{0,j_1}(T_X^{1,0}) \otimes \Omega^{0,j_2}(T_X^{1,0}) \to \Omega^{0,j_1+j_2+1}(T_X^{1,0})$$

  induced by the Dolbeault representative R_2 of the Atiyah class;
- for every $k \geqslant 3$, the k-th multibracket λ_k is the composition of the wedge product $\Omega^{0,j_1}(T_X^{1,0}) \otimes \cdots \otimes \Omega^{0,j_n}(T_X^{1,0}) \to \Omega^{0,j_1+\cdots+j_k}\big((T_X^{1,0})^{\otimes k}\big)$ with the map $\Omega^{0,j_1+\cdots+j_k}\big((T_X^{1,0})^{\otimes k}\big) \to \Omega^{0,j_1+\cdots+j_n+1}(T_X^{1,0})$ induced by an element R_k of $\Omega^{0,1}\big(S^k(T_X^{1,0})^\vee \otimes T_X^{1,0}\big)$ arising as an algebraic function of R_2, the curvature of $\nabla^{1,0}$, and their higher covariant derivatives.

Atiyah class of a dg manifold

Lemma:

- $\mathcal{L}_Q R_2 = \mathcal{L}_Q(\mathcal{L}_Q \nabla) = 0$
- $[R_2] = [\mathcal{L}_Q \nabla] \in H^1\big(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), \mathcal{L}_Q\big)$

  is independent of the connection ∇

Definition: The Atiyah class of the dg manifold $(\mathcal{M}, Q)$ is

$$\alpha_{\mathcal{M}} := [R_2] \in H^1\big(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), \mathcal{L}_Q\big).$$

It is the obstruction to existence of an affine connection ∇ on $\mathcal{M}$ compatible with the homological vector field Q in the sense that

$$\mathcal{L}_Q(\nabla_X Y) = \nabla_{\mathcal{L}_Q X} Y + (-1)^{|X|} \nabla_X(\mathcal{L}_Q Y) \quad \text{for all } X, Y \in \mathfrak{X}(\mathcal{M}).$$

- Lyakhovich, Mosman, Sharapov
- Mehta, S, Xu

Example: dg manifold $(\mathbb{R}^{m|n}, Q)$

- $(x_1, \cdots, x_m; x_{m+1} \cdots x_{m+n})$ are coordinate functions on $\mathbb{R}^{m|n}$
- $Q = \sum_k Q_k(x) \frac{\partial}{\partial x_k}$
- trivial connection $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$
- $\alpha_{\mathbb{R}^{m|n}} \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = (-1)^{|x_i|+|x_j|} \sum_k \frac{\partial^2 Q_k}{\partial x_i \partial x_j} \frac{\partial}{\partial x_k}$

Example: $\mathfrak{g}$ is a finite-dimensional Lie algebra

- $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ is corresponding dg manifold
- $T_{\mathcal{M}} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]$ implies

$$H^1\big(\Gamma(S^2(T^\vee_{\mathcal{M}}) \otimes T_{\mathcal{M}}), \mathcal{L}_Q\big) \cong H^0_{\mathrm{CE}}(\mathfrak{g}; \Lambda^2\mathfrak{g}^\vee \otimes \mathfrak{g}) \cong (\Lambda^2\mathfrak{g}^\vee \otimes \mathfrak{g})^{\mathfrak{g}}$$

- $\alpha_{\mathfrak{g}[1]} \in (\Lambda^2\mathfrak{g}^\vee \otimes \mathfrak{g})^{\mathfrak{g}}$ is precisely the Lie bracket of $\mathfrak{g}$

Todd class of a dg manifold

$$\mathrm{Td}_{\mathcal{M}} := \mathrm{Ber}\left(\frac{1-e^{-\alpha_{\mathcal{M}}}}{\alpha_{\mathcal{M}}}\right) \in \prod_{k\geqslant 0} H^k\big(\Omega^k(\mathcal{M}), \mathcal{L}_Q\big)$$

Example: Every Lie algebra $\mathfrak{g}$ determines a dg manifold

$$(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}}).$$

- $H^k\big(\Omega^k(\mathcal{M}), \mathcal{L}_Q\big) \cong H^0_{\mathrm{CE}}(\mathfrak{g}; S^k\mathfrak{g}^\vee) \cong \big(S^k(\mathfrak{g}^\vee)\big)^{\mathfrak{g}}$
- $\prod_{k\geqslant 0} H^k\big(\Omega^k(\mathcal{M}), \mathcal{L}_Q\big) \ni \mathrm{Td}_{\mathcal{M}} \longmapsto \det\left(\frac{1-e^{-\mathrm{ad}}}{\mathrm{ad}}\right) \in \big(\hat{S}(\mathfrak{g}^\vee)\big)^{\mathfrak{g}}$

(Duflo element of $\mathfrak{g}$)

Theorem: If the Atiyah class $\alpha_{\mathcal{M}}$ vanishes, then there exists a torsionfree connection such that

$$\Gamma(S(T_{\mathcal{M}})) \xrightarrow{\mathrm{pbw}} \mathcal{D}(\mathcal{M})$$

is an isomorphism of dg coalgebras over $C^\infty(\mathcal{M})$, i.e. $\mathcal{L}_Q \circ \mathrm{pbw} = \mathrm{pbw} \circ \mathcal{L}_Q$

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Dg vector bundles

Definition: A dg vector bundle is a vector bundle object in the category of dg manifolds.

Suppose $\mathcal{E} \to \mathcal{M}$ is a vector bundle object in the category of $\mathbb{Z}$-graded manifolds and $\mathcal{M}$ admits a homological vector field Q. Then $\mathcal{E}$ admits a dg manifold structure making $\mathcal{E} \to \mathcal{M}$ into a dg vector bundle if and only if $\Gamma(\mathcal{E})$ admits a structure of dg module over the dg algebra $\big(C^\infty(\mathcal{M}), Q\big)$.

Indeed, the category of dg vector bundles over the dg manifold $(\mathcal{M}, Q)$ is equivalent to the category of locally free dg modules over the dg Lie algebra $\big(C^\infty(\mathcal{M}), Q\big)$.

Example: Let $\mathfrak{g}$ be a f.d. Lie algebra and let V be a f.d. vector space. A structure of $\mathfrak{g}$-module on V is equivalent to a structure of dg vector bundle on $\mathfrak{g}[1] \times V \to \mathfrak{g}[1]$.

Example: Given an L_∞ algebra $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i$, saying that a $\mathbb{Z}$-graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$ is an L_∞ module over $\mathfrak{g}$ is equivalent to saying that $\mathfrak{g}[1] \times V \to \mathfrak{g}[1]$ is a dg vector bundle.

Example:

- Let X be a complex manifold.
- Let $E \to X$ be a complex vector bundle.
- Let π^*E denote the pullback of the complex vector bundle $E \to X$ through the canonical projection $\pi : T_X^{0,1}[1] \to X$.

Then $E \to X$ is a holomorphic vector bundle iff $\pi^*E \to T_X^{0,1}[1]$ is a dg vector bundle.

Dg Lie algebroids

Definition: A dg Lie algebroid is a Lie algebroid object in the category of dg manifolds.

More precisely, a dg Lie algebroid consists of

- a dg vector bundle $\mathcal{A} \to \mathcal{M}$
- together with a vector bundle map $\rho : \mathcal{A} \to T_{\mathcal{M}}$ of degree 0 called anchor and a graded Lie algebra structure on $\Gamma(\mathcal{A})$ with Lie bracket satisfying

$$[X, fY] = \rho_X(f)Y + (-1)^{|X||f|} f\,[X, Y]$$

  for all homogeneous $X, Y \in \Gamma(\mathcal{A})$ and $f \in C^\infty(\mathcal{M})$

The dg and Lie structures must be compatible: $\boxed{[\mathcal{Q}, d_{\mathcal{A}}] = 0}$, where

- $\mathcal{Q} \in \mathfrak{X}(\mathcal{A}[1])$ is the homological v.f. on $\mathcal{A}[1]$ induced by the homological v.f. on total space $\mathcal{A}$ of dg v.b. structure
- and $d_{\mathcal{A}} \in \mathfrak{X}(\mathcal{A}[1])$ is the Chevalley–Eilenberg differential arising from the Lie algebroid structure.

Proposition (S, Xu):

- Let $\mathcal{A} \to \mathcal{M}$ be a Lie algebroid object in the category of $\mathbb{Z}$-graded manifolds with anchor map $\rho : \mathcal{A} \to T_{\mathcal{M}}$
- and let $s \in \Gamma(\mathcal{A})$ be a section of degree $+1$ satisfying $[s, s] = 0$.
- Then $\mathcal{A} \to \mathcal{M}$ admits a structure of *dg* Lie algebroid:
  - the homological v.f. on $\mathcal{M}$ is $\rho(s)$
  - while the operator of degree $+1$ on $\Gamma(\mathcal{A})$ is $[s, -]$.

Example (S, Vitagliano, Xu):

- Let $\phi : A \to L$ be a morphism of Lie algebroids (with base M).
- Pulling back (in the Lie algebroid sense) the Lie algebroid $L \to M$ through $A[1] \twoheadrightarrow M$ yields the Lie algebroid (object in the category $\mathbb{Z}$-graded manifolds) $T_{A[1]} \times_{T_M} L \to A[1]$.
- Together, the vector field $d_{\mathcal{A}} \in \mathfrak{X}(\mathcal{A}[1])$ and the map $A[1] \to A \xrightarrow{\phi} L$ determine a section s_ϕ of $T_{A[1]} \times_{T_M} L \to A[1]$ of degree $+1$ and satisfying $[s_\phi, s_\phi] = 0$.
- Proposition above $\Longrightarrow T_{A[1]} \times_{T_M} L \to A[1]$ is a dg Lie algebroid.

Atiyah class of a dg v.b. relative to a dg Lie alg'oid

- dg vector bundle $\mathcal{E} \to \mathcal{M}$ — special case: $\mathcal{E} = \mathcal{A} = T_{\mathcal{M}}$
- dg Lie algebroid $\mathcal{A} \to \mathcal{M}$ — special case: $\mathcal{E} = \mathcal{A} \neq T_{\mathcal{M}}$
- Choose an $\mathcal{A}$-connection on $\mathcal{E}$, i.e. a map of degree 0

$$\nabla : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$$

  satisfying $\nabla_{fX} s = f \nabla_X s$ and $\nabla_X(fs) = \rho_X(f)s + (-1)^{|X||f|} \nabla_X s$.
- Consider bundle map $\operatorname{At}^\nabla : \mathcal{A} \otimes \mathcal{E} \to \mathcal{E}$ of degree $+1$ defined by

$$\operatorname{At}^\nabla(X, s) = \mathcal{Q}(\nabla_X s) - \nabla_{\mathcal{Q}(X)} s - (-1)^{|X|} \nabla_X \big(\mathcal{Q}(s)\big).$$

- Fact: $\operatorname{At}^\nabla \in \Gamma(\mathcal{A}^\vee \otimes \operatorname{End} \mathcal{E})$ is a 1-cocycle: $\mathcal{Q}(\operatorname{At}^\nabla) = 0$.
- Its cohomology class $\alpha = [\operatorname{At}^\nabla] \in H^1\big(\Gamma(\mathcal{A}^\vee \otimes \operatorname{End} \mathcal{E})^\bullet, \mathcal{Q}\big)$ is independent of the choice of ∇.
- This class α is called Atiyah class of the dg v.b. $\mathcal{E}$ relative to the dg Lie algebroid $\mathcal{A}$.

Thank You

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