Hopf monads, Hopf algebras and diagrammatics

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Structure of the talk

1. Motivation
2. Monads
3. Bimonads
4. Hopf monads
5. Augmented Hopf monads

1: Motivation

Motivation stuff

X complex manifold, $D(X)$ is a symmetric monoidal category with duals.

Atiyah class gives Lie algebra $T[-1] \in D(X)$ acting on all objects in $D(X)$.

Using the diagonal $\Delta\colon X \to X \times X$ get adjoint functors

$$\Delta_*\colon D(X) \leftrightarrows D(X \times X) :\Delta^!.$$

Define $\mathcal{U} := \Delta^! \Delta_* \mathcal{O}_X$.

Then $\mathcal{U}$ is the universal enveloping algebra of $T[-1]$.

Also have $\mathcal{U} = \pi_* \mathcal{H}\mathrm{om}(\Delta_* \mathcal{O}_X, \Delta_* \mathcal{O}_X)$.

So $\mathcal{U}$ is an associative algebra which acts on everything in the category.

- Is $\mathcal{U}$ a Hopf algebra?
- Does $\mathcal{U} \in D(X)$ behave like $\mathbb{C}G^{\mathrm{ad}} \in \mathrm{Rep}(G)$ for G a finite group?

Reconstruction

For $\mathcal{C}$ a monoidal category with duals the end construction

$$E := \int_{V \in \mathcal{C}} V^{\vee} \otimes V$$

gives (if it exists) a Hopf algebra which acts on every object in the category.

E.g. if $\mathcal{C} = \mathsf{Rep}(G)$ for G a finite group then $E = \mathbb{C}G^{\mathrm{ad}}$.

However, $D(X)$ does not have enough limits and the end does not exist.

Try a different tack.

For a finite group G the diagonal $\Delta\colon G \to G \times G$ gives adjoint functors

$$\Delta_!\colon \mathsf{Rep}(G) \leftrightarrows \mathsf{Rep}(G \times G) : \Delta^*$$

with $\mathbb{C}G^{\mathrm{bi}} \cong \Delta^* \Delta_! \mathbb{C}$.

- Can we use properties of Δ^* and $\Delta_!$ to show $\Delta^* \Delta_! \mathbb{C}$ is a Hopf algebra?

2: Monads

Monads definition

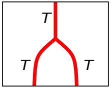
For $\mathcal{C}$ a category a monad $T\colon \mathcal{C} \to \mathcal{C}$ is an algebra in $(\text{End}(\mathcal{C}), \circ, \mathrm{id})$.

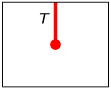
This amounts to

- endofunctor $T\colon \mathcal{C} \to \mathcal{C}$
- a product $\mu\colon T \circ T \Rightarrow T$
- unit $\iota\colon \mathrm{id} \Rightarrow T$

satisfying associativity and unit axioms.

We draw these as follows.

$\mu \equiv$ 

and $\iota \equiv$ 

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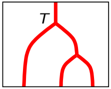
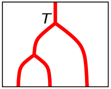
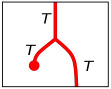
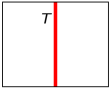
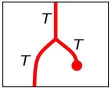
These have to satisfy the associativity and unit laws, namely



$=$ 

and 

$=$ 

$=$ 

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Example 1: monads from algebras

Suppose $(\mathcal{C}, \otimes, \mathbb{1})$ is a monoidal category and A is an algebra in $\mathcal{C}$.

Then $(A \otimes -)\colon \mathcal{C} \to \mathcal{C}$ is a monad.

E.g. For any X in $\mathcal{C}$, the product

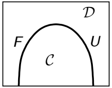
$$\mu_X\colon A \otimes A \otimes X \to A \otimes X$$

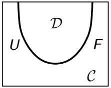
comes, in the obvious way, from the algebra product on A.

Example 2: monads from adjunctions

Suppose that $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ forms an adjunction.

The counit $\epsilon\colon F \circ U \Rightarrow \mathrm{id}_{\mathcal{D}}$ and unit $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow U \circ F$ are drawn as follows.

$\epsilon \equiv$ 

; $\eta \equiv$ 

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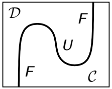
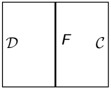
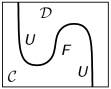
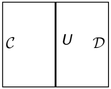
The required conditions on the unit and counit are drawn as



$=$ 

and 

$=$ 

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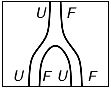
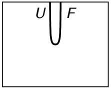
Then $U \circ F\colon \mathcal{C} \to \mathcal{C}$ forms a monad.

The multiplication and unit are formed from the unit and counit as follows:

$\mu \equiv$ 

; $\iota \equiv$ 

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Category of modules

A monad $T\colon \mathcal{C} \to \mathcal{C}$ has a category of modules $\mathcal{C}^T$.

The objects are pairs $(X \in \mathcal{C},\ \rho\colon T(X) \to X)$.

The morphisms are morphisms in $\mathcal{C}$ commuting with the actions.

E.g. For $T = (A \otimes -)$ then $\mathcal{C}^T$ is the usual category of modules of A.

3: Bimonads

Bialgebras and tensoring modules

Suppose $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category (e.g. symmetric).

Then having a bialgebra structure on an algebra A means we can put a monoidal structure on the tensor product of two modules:

$$A \otimes X \otimes Y \to A \otimes A \otimes X \otimes Y \to A \otimes X \otimes A \otimes Y \to X \otimes Y.$$

More precisely we can say that $\otimes$ lifts from $\mathcal{C}$ to $\mathrm{Rep}(A)$.

Lifting the tensor product for monads

Suppose we have a monad $T\colon \mathcal{C} \to \mathcal{C}$ on a monoidal category.

If we want to lift $\otimes$ from $\mathcal{C}$ to $\mathcal{C}^T$ then we cannot do it by thinking of bialgebras in $\mathrm{End}(\mathcal{C})$ as this does not have a braiding, even if $\mathcal{C}$ does.

In general, for $F_1, F_2\colon \mathcal{C} \to \mathcal{C}$

$$F_1 \circ F_2 \ncong F_2 \circ F_1.$$

Theorem (Moerdijk)

Suppose $(C, \otimes, \mathbb{1})$ is a monoidal category and $T\colon \mathcal{C} \to \mathcal{C}$ is a monad. Lifts of $\otimes$ from $\mathcal{C}$ to $\mathcal{C}^T$ corresponds to opmonoidal structures on T.

For this reason, call a monad with an opmonoidal structure a bimonad.

Opmonoidal monads, a.k.a. bimonads

Suppose $(\mathcal{C}, \otimes, \mathbb{1})$ is monoidal category.

A monad T on $\mathcal{C}$ is opmonoidal if we have (not necessarily isomorphisms):

$$T(\mathbb{1}) \to \mathbb{1} \quad \text{and} \quad T(X \otimes Y) \to T(X) \otimes T(Y) \quad \text{for } X, Y \in \mathcal{C},$$

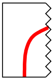
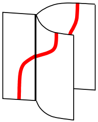
in a way compatible with the associativity and unitality of the monoidal structure and with the product and unit of the monad.

More precisely, an opmonoidal structure on T consists of natural transformations (obeying associativity and unitality conditions)

$$\sigma_2^T\colon T \circ \mathbb{1} \Rightarrow \mathbb{1} \quad \text{and} \quad \sigma_2^T\colon T \circ \otimes \Rightarrow \otimes \circ (T \times T)$$

which commute with the product μ and unit ι.

Diagrammatically:

$\sigma_0^T \equiv$ , $\qquad \sigma_2^T \equiv$ .

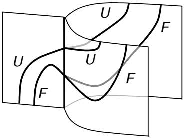
Examples of bimonads

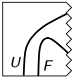
Example

Example: Suppose $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category.
If A is a bialgebra then $(A \otimes -) \colon \mathcal{C} \to \mathcal{C}$ is a bimonad in an obvious way.

Example

Suppose $\mathcal{C}$ and $\mathcal{D}$ are monoidal categories.
Given an adjunction $F \colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ with U strong monoidal, then the monad $UF \colon \mathcal{C} \to \mathcal{C}$ is canonically a bimonad with

$\sigma_2^{UF} \equiv$ 

, and $\sigma_0^{UF} \equiv$ 

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Example

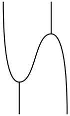
We have the adjunction $\Delta_! \colon \operatorname{Rep}(G) \leftrightarrows \operatorname{Rep}(G \times G) \colon \Delta^*$.
Pull backs are strong monoidal so $\Delta^* \Delta_!$ is a bimonad.

4: Hopf monads

Hopf algebras

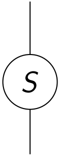
Hopf algebras are bialgebras with a certain property.
[Ignore left and right differences in this talk.]

For a bialgebra A define the fusion operator $V\colon A \otimes A \to A \otimes A$ as

$$V := (\mathrm{id} \otimes \mu) \circ (\delta \otimes \mathrm{id}) =$$  .

Theorem (Street?)

A bialgebra A is a Hopf algebra if and only if the fusion operator is invertible.



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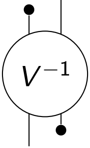
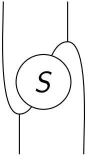


; $V^{-1} =$


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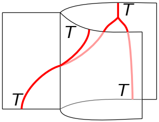
Hopf monads (Bruguières, Lack, Virelizier)

For a bimonad T on a monoidal category, the fusion operator

$$H\colon T \circ \otimes \circ (\mathrm{id} \times T) \Rightarrow \otimes \circ (T \times T)$$

is defined via



A bimonad is a Hopf monad if the fusion operator is invertible.

Theorem (BLV)

Suppose $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category with duals. If T is a Hopf monad on $\mathcal{C}$ then $\mathcal{C}^T$ has duals.

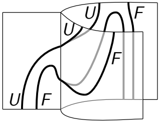
Example

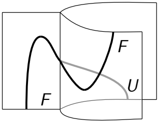
If $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$ is a braided monoidal category and A is a Hopf monad then the bimonad $(A \otimes -)$ is a Hopf monad.

Examples of Hopf monads

Example

Given an adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ with U strong monoidal, then

$H :=$ 

is invertible $\iff$ $\mathbb{H} :=$ 

is invertible

In other words, if and only if we have natural isomorphisms

$$F(X \otimes U(Y)) \xrightarrow{\sim} F(X) \otimes Y \quad \text{for all } X \in \mathcal{C}, Y \in \mathcal{D}.$$

In this case we say the projection formula holds.
So UF is a Hopf monad if and only if the projection formula holds.

Example

A classic example is for $f\colon G \to K$ finite group homomorphism then

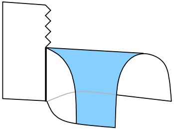
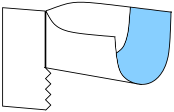
$$f_!(V \otimes f^*W) \cong f_! V \otimes W \quad \text{for all } V \in \mathrm{Rep}(G), W \in \mathrm{Rep}(K).$$

Categories with duals

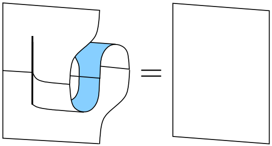
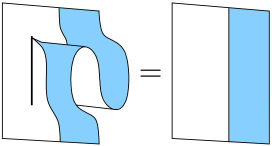
If we have a monoidal category $\mathcal{C}$ with a duals, then we have ${}^\vee\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$ with coevaluation and evaluation maps.

$$\mathrm{ev}\colon X^\vee \otimes X \to \mathbb{1} \quad \text{and} \quad \mathrm{coev}\colon \mathbb{1} \to X \otimes X^\vee.$$

These give rise to so-called dinatural transformations which can be drawn as

ev $\equiv$  coev $\equiv$ .

These satisfy the so-called snake relations which become the following:

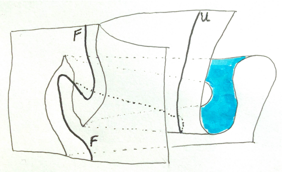
 ;  .

The projection formula

Theorem (Fausk, Hu, May?)

Given an adjunction $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ with U strong monoidal, such that $\mathcal{C}$ and $\mathcal{D}$ have duals then the projection formula holds, so UF is a Hopf monad.

We can write down the inverse of the Hopf operator explicitly in this case.

$\mathbb{H}^{-1} =$ 

So $\Delta^*\Delta_!$ is a Hopf monad — for both $\mathrm{Rep}(G)$ and $D(X)$.

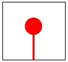
Question: Is $\Delta^*\Delta_! \cong (A \otimes -)$, as Hopf monads, for a Hopf algebra A?

In that case we would have $A = \Delta^*\Delta_!(\mathbb{1})$.

5: Augmented Hopf monads

Augmentations

An augmentation of a Hopf monad T is a bimonad map $e\colon T \to \mathrm{id}$.



An augmentation is the same as an action on each object of the category:

$$e_X\colon T(X) \to X \quad \text{for all } X.$$

And in fact gives a functor $\mathcal{C} \to \mathcal{C}^T$.

Theorem (BLV)

Let T be a Hopf monad on a braided monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, \tau)$.

$$T \cong (T(\mathbb{1}) \otimes -) \text{ as Hopf monads, with } T(\mathbb{1}) \text{ a Hopf algebra}$$
$$\iff \exists e\colon T \to \mathrm{id} \text{ an augmentation compatible with the braiding } \tau.$$

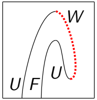
[If T is braided opmonoidal then all augmentations are compatible with τ.]

Augmentations from right inverses

Now we will use the fact that the diagonal map Δ has a one sided inverse.

Theorem

If $\mathcal{C}$ and $\mathcal{D}$ are braided monoidal with duals, $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : U$ is an adjunction with U strong monoidal and U has a right inverse W, then UF has an augmentation $e\colon UF \to \mathrm{id}$ and $UF \cong (UF(\mathbb{1}) \otimes -)$ as Hopf monads.



For the adjunctions

$$\Delta_!\colon \operatorname{Rep}(G) \leftrightarrows \operatorname{Rep}(G \times G)\colon \Delta^* \quad \text{and} \quad \Delta_!\colon D(X) \leftrightarrows D(X \times X)\colon \Delta^*$$

we can take $W = \pi_1^*$.

The payoff

We thus find that both

$$\Delta^*\Delta_!(\mathbb{C}) \cong \mathbb{C}G^{\mathrm{ad}} \in \mathsf{Rep}(G)$$

and

$$\Delta^*\Delta_!(\mathcal{O}_X) \cong \left(\Delta^*\Delta_*(\mathcal{O}_X)\right)^\vee \cong \Delta^!\Delta_*(\mathcal{O}_X) = \mathcal{U} \in D(X)$$

are Hopf algebras which act on the objects in their respective categories.

We will be able to write down the structure explicitly.

However, not clear we have the all of the right structure yet.

We might need another augmentation.