Atiyah classes and Hochschild cohomology of integrable distributions

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January 8, 2020

Joint work with Z.Chen and P.Xu.
Workshop on Atiyah classes and related topics, KIAS.

A motivating theorem

Theorem (Kontsevich, Calaque-Van den Bergh, Liao-Stiénon-Xu)

Let X be a complex manifold. The composition

$$\mathrm{hkr} \circ \mathrm{Td}^{1/2}_{T_X^{\mathbb{C}}/T_X^{0,1}} : H(X, \wedge^\bullet T_X) \xrightarrow{\cong} HH^\bullet(X)$$

is an isomorphism of Gerstenhaber algebras, where

- $\mathrm{hkr} : H^\bullet(X, \wedge^\bullet T_X) \to HH^\bullet(X)$ *is the Hochschild-Konstant-Rosenberg isomorphism of graded vector spaces derived by Gerstenhaber and Schack (also cf. Căldăraru's work);*
- $\mathrm{Td}_{T_X^{\mathbb{C}}/T_X^{0,1}}$ *is the Todd class of the Lie algebroid pair $(T_X^{\mathbb{C}}, T_X^{0,1})$ introduced by Chen, Stiénon and Xu, which is isomorphic to the Todd class Td_X of X when the Hodge decomposition holds.*

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- Kontsevich proved the above isomorphism only as associative algebras;
- Calaque-Van den Bergh proved the above theorem for any smooth algebraic variety.
- Liao-Stiénon-Xu proved the above theorem for any complex manifold.

Goal of this talk

The goal of this talk is to explain the previous theorem in the text of dg geometry via the following commutative diagram in our poster

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[-1], \mathcal{Q}) & \xrightarrow[\cong]{\mathrm{hkr}\circ \mathrm{Td}^{1/2}_{(\mathcal{M},Q)}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[-1], \mathcal{Q}+d_{\mathscr{H}}) \\ \Phi^{\bullet,0}\big\downarrow\cong & & \cong\big\downarrow\Phi^\bullet_\natural \\ H^\bullet(X, \wedge^\bullet T_X) & \xrightarrow[\cong]{\mathrm{hkr}\circ \mathrm{Td}^{1/2}_{T_X^{\mathbb{C}}/T_X^{0,1}}} & HH^\bullet(X), \end{array}$$

where $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ is the dg manifold from the complex manifold X.

Outline

Polyvector fields on dg manifolds

$\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$: a smooth $\mathbb{Z}$-graded manifold; $\mathcal{R} = \mathcal{O}_{\mathcal{M}}(M)$.
$(\mathcal{T}^p_{\mathrm{poly}}(\mathcal{M}))^q = (\Gamma(\wedge^{p+1} T_{\mathcal{M}}))^q$: the space of $(p+1)$-vector fields of degree q on $\mathcal{M}$.
Consider the graded left $\mathcal{R}$-module $\mathrm{tot}^{\bullet}_{\oplus}(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})) = \oplus_n \mathrm{tot}^n_{\oplus}(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))$ defined by

$$\mathrm{tot}^n_{\oplus}(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})) = \bigoplus_{p \geq -1, p+q=n} \left(\mathcal{T}^p_{\mathrm{poly}}(\mathcal{M})\right)^q = \bigoplus_{p \geq -1, p+q=n} \left(\Gamma(\wedge^{p+1} T_{\mathcal{M}})\right)^q .$$

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- $\text{tot}^\bullet_\oplus(\mathcal{T}_{\text{poly}}(\mathcal{M}))[-1]$, together with the wedge product $\wedge$ and the Schouten bracket $[-,-]$, is a Gerstenhaber algebra.
- Given each homological vector field $Q \in (\mathcal{T}^0_{\text{poly}}(\mathcal{M}))^1$ on $\mathcal{M}$, being a Maurer-Cartan element of the dg Lie algebra $(\text{tot}^\bullet_\oplus(\mathcal{T}_{\text{poly}}(\mathcal{M})), [-,-])$, we have the tangent dg Lie algebra at Q $(\text{tot}^\bullet_\oplus(\mathcal{T}_{\text{poly}}(\mathcal{M})), [Q,-], [-,-])$, and a Gerstenhaber algebra on the cohomology

$$(H^\bullet(\text{tot}_\oplus(\mathcal{T}_{\text{poly}}(\mathcal{M}))[-1], [Q,-]), \wedge, [-,-]).$$

Polydifferential operators on dg manifolds

A differential operator $D \in \mathcal{D}(\mathcal{M})^q$ of degree q on a graded manifold $\mathcal{M}$ has the form of a finite sum $D = \sum X_1 \circ \cdots \circ X_k$ of compositions of graded derivations $X_1, \cdots, X_k$ of $\mathcal{R}$ with $\sum_{i=1}^{k} |X_i| = q$.

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- Each homological vector field $Q \in \mathcal{D}(\mathcal{M})^1 = (\mathcal{D}_{\mathrm{poly}}^{0}(\mathcal{M}))^1$ is a Maurer-Cartan element of this dg Lie algebra.

$$(\operatorname{tot}_{\oplus}^{\bullet}(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})))_Q = (\operatorname{tot}_{\oplus}^{\bullet}(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), [Q,-]_G + d_{\mathscr{H}}, [-,-]_G).$$

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- We have a Gerstenhaber algebra on the cohomology level

$$(H^\bullet(\operatorname{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[-1], [Q,-]_G + d_{\mathscr{H}}), [-,-]_G, \cup = \otimes_{\mathcal{R}}).$$

Hochschild-Konstant-Rosenberg isomorphism for dg manifolds

For any graded manifold $\mathcal{M}$, the Hochschild-Konstant-Rosenberg map is defined, as usual, to be the natural inclusion by skew-symmetrization

$$\mathrm{hkr} : \mathrm{tot}^\bullet_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})) \hookrightarrow \mathrm{tot}^\bullet_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))$$

$$\mathrm{hkr}(X_0 \wedge \cdots \wedge X_p) = \frac{1}{(p+1)!} \sum_{\sigma \in S_{p+1}} \kappa(\sigma) X_{\sigma(0)} \otimes \cdots \otimes X_{\sigma(p)},$$

for all homogeneous vector fields $X_0, \cdots, X_p \in (\mathcal{T}^0_{\mathrm{poly}}(\mathcal{M}))^\bullet$, where the skew-Koszul sign $\kappa(\sigma)$ is the scalar defined by the relation

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Proposition (Liao-Stiénon-Xu)

For any finite dimensional dg manifold $(\mathcal{M}, Q)$, the map hkr induces an isomorphism of graded vector spaces on the cohomology level

$$\mathrm{hkr} : H^{\bullet}(\mathrm{tot}^{\bullet}_{\oplus}(\mathcal{T}_{\mathrm{poly}}(\mathcal{M})), [Q, -]) \xrightarrow{\cong} H^{\bullet}(\mathrm{tot}^{\bullet}_{\oplus}(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), [Q, -]_G + d_{\mathscr{H}}).$$

Atiyah classes of dg manifolds

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$$\alpha_{\mathcal{M}}^{\nabla}(X, Y) = L_Q(\nabla_X Y) - \nabla_{L_Q(X)} Y - (-1)^{|X|} \nabla_X L_Q(Y),$$

for all $X, Y \in \Gamma(T_{\mathcal{M}})$, which is L_Q-closed. Its cohomology class

$$\alpha_{\mathcal{M}} = [\alpha_{\mathcal{M}}^{\nabla}] \in H^1(\Gamma(T_{\mathcal{M}}^{\vee} \otimes \mathrm{End}(T_{\mathcal{M}})), L_Q)$$

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- (Lyakhovich, Mosman and Sharapov) The covariant derivative $\Lambda := \nabla(Q) \in \Gamma(\mathrm{End}(T_{\mathcal{M}}))$ of Q determines also determines a $(1,2)$-tensor $B_1 \in \Gamma(\mathrm{Hom}(T_{\mathcal{M}}, \mathrm{End}(T_{\mathcal{M}})))$ by

$$B_1(X) = (-1)^{|X|}(\nabla_X \Lambda - R^{\nabla}(X, Q)),$$

  where $R^{\nabla}(X, Q) = [\nabla_X, \nabla_Q] - \nabla_{[X,Q]}$ is the curvature tensor of ∇. Its cohomology class $[B_1]$ is, in fact, the first type B stable characteristic class of $(\mathcal{M}, Q)$.

Todd class and Kontsevich-Duflo type isomorphism for dg manifolds

The Todd class of a dg manifold $(\mathcal{M}, Q)$ is defined by

$$\mathrm{Td}_{(\mathcal{M},Q)} := \mathrm{Ber}\left(\frac{\alpha_{\mathcal{M}}}{1 - e^{-\alpha_{\mathcal{M}}}}\right) \in \prod_{k \geq 0} H^k\left((\Omega^k(\mathcal{M}))^{\bullet}, L_Q\right),$$

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Theorem (Liao-Stiénon-Xu)

For any finite dimensional dg manifold $(\mathcal{M}, Q)$, the composition

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of the action of the square root $\mathrm{Td}^{1/2}_{(\mathcal{M},Q)} \in \prod_k H^k((\Omega^k(\mathcal{M}))^\bullet, \mathcal{Q})$ of the Todd class of $(\mathcal{M}, Q)$, by contraction, with the Hochschild-Konstant-Rosenberg map hkr is an isomorphism of Gerstenhaber algebras.

The Atiyah and Todd classes in complex geometry

X: a complex manifold, T_X: its tangent sheaf, and $T_X^{\mathbb{C}}$ its the complexified tangent bundle with the decomposition $T_X^{\mathbb{C}} = T_X^{1,0} \oplus T_X^{0,1}$.

- The Atiyah class

$$\alpha_X \in H^1_{\mathrm{Dol}}(X, (T_X^{1,0})^\vee \otimes \mathrm{End}(T_X^{1,0})) \cong H^1(X, T_X^\vee \otimes \mathrm{End}(T_X))$$

is the obstruction to existence of holomorphic connections on $T_X^{1,0}$.

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$$\mathrm{Td}_X = \prod_i \frac{\alpha_i}{1 - e^{-\alpha_i}} \in \oplus_{k\geq 0} H^{2k}(X, \mathbb{C}),$$

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- In particular, when the Hodge decomposition holds for X, one has $\mathrm{tr}(\alpha_X) = c_1(X) \in H^{1,1}(X) \subset H^2(X, \mathbb{C})$, and the Todd class Td_X may be recovered by the Atiyah class α_X via

$$\mathrm{Td}_X = \mathrm{Td}_{T_X^{\mathbb{C}}/T_X^{0,1}} = \det\left(\frac{\alpha_X}{1 - e^{-\alpha_X}}\right) \in \bigoplus_{k \geq 0} H^{k,k}(X, \mathbb{C}) \subset \bigoplus_{k \geq 0} H^{2k}(X, \mathbb{C}),$$

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Atiyah and Todd classes of associated dg manifolds

Note that each complex manifold gives rise to a dg manifold

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We have the Atiyah class and Todd class of the associated dg manifold

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Question

What is the relation between the two types of Atiyah and Todd classes?

Cohomology of dg manifolds from complex manifolds

Let $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ be the dg manifold from a complex manifold X.

Proposition

There is a contraction of the space $\Gamma(T_\mathcal{M})$ of vector fields on $\mathcal{M}$

$$h \circlearrowright (\Gamma(T_\mathcal{M}), [\bar{\partial}, -]) \underset{\psi}{\overset{\phi}{\rightleftarrows}} (\Omega_X^{0,\bullet}(T_X^{1,0}), \bar{\partial}),$$

where ϕ, ψ, h are all $\Omega_X^{0,\bullet}$-linear.

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where ϕ, ψ, h are all $\Omega_X^{0,\bullet}$-linear.

Their dual maps give rise to a contraction on the space $\Omega^1(\mathcal{M})$ of 1-forms on $\mathcal{M}$

$$h^\vee \circlearrowright (\Omega^1(\mathcal{M}), L_{\bar{\partial}}) \underset{\psi^\vee}{\overset{\phi^\vee}{\rightleftarrows}} (\Omega_X^{0,\bullet}((T_X^{1,0})^\vee), \bar{\partial}).$$

Isomorphisms between two types of Atiyah and Todd classes

Applying the tensor trick on contractions, we obtain a contraction for all (p, q)-tensor fields on $(\mathcal{M}, Q)$

$$H^{p,q} \circlearrowright (\Gamma((T_{\mathcal{M}})^{\otimes p} \otimes (T_{\mathcal{M}}^{\vee})^{\otimes q}), L_{\bar{\partial}}) \underset{\Psi^{p,q}}{\overset{\Phi^{p,q}}{\rightleftarrows}} (\Omega_X^{0,\bullet}((T_X^{1,0})^{\otimes p} \otimes ((T_X^{1,0})^{\vee})^{\otimes q}), \bar{\partial}).$$

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Theorem (Chen-Xiang-Xu)

Let $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ *be the dg manifold from a complex manifold* X*, there exist canonical isomorphisms*

$$\Phi^{m,n} : H^{\bullet}(\Gamma((T_{\mathcal{M}})^{\otimes m} \otimes (T_{\mathcal{M}}^{\vee})^{\otimes n}), L_Q) \xrightarrow{\cong} H^{\bullet}(X, (T_X)^{\otimes m} \otimes (T_X^{\vee})^{\otimes n})$$

from the cohomology of (m,n)*-tensor fields on the dg manifold* $(\mathcal{M}, Q)$ *to the sheaf cohomology of* X *such that*

- $\Phi^{1,2}(\alpha_{\mathcal{M}}) = \alpha_X$;
- $\Phi^{0,\bullet}(\mathrm{Td}_{\mathcal{M}}) = \mathrm{Td}_{T_X^{\mathbb{C}}/T_X^{0,1}}$, *which is further isomorphic to the Todd class* Td_X *of* X *when the Hodge decomposition holds for* X *(e.g., compact Kähler or smooth algebraic).*

Polyvector fields on complex manifolds

X: complex manifold, T_X its tangent sheaf.

- The sheaf cohomology $H^\bullet(X, \wedge^\bullet T_X)$, together with the wedge product $\wedge$ and the Schouten bracket $[-,-]$, is a Gerstenhaber algebra.

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Hochschild cohomology of complex manifolds

X: a complex manifold. Its Hochschild cohomology is

$$HH^\bullet(X) := \mathrm{Ext}^\bullet_{\mathcal{O}_{X\times X}}(\mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

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Let $\mathcal{D}^\bullet_{\mathrm{poly}}(X) \cong (\mathcal{U}(T_X^{1,0}))^{\otimes \bullet+1}$ be left module of holomorphic differential operators on X. The hypercohomology

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The cohomology of polydifferential operators on $(\mathcal{M}, Q) = (T^{0,1}_X[1], \bar{\partial})$

$$(H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[-1], [Q, -]_G + d_{\mathscr{H}}), [-,-]_G, \cup)$$

is also a Gerstenhaber algebra.

Coalgebra contraction for the polydifferential operators

Proposition (Chen-Xiang-Xu)

There is an $\Omega_X^{0,\bullet}$-coalgebra contraction

$$\breve{H}_\natural \circlearrowright (\mathrm{tot}_{\oplus}^{\bullet}(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), [Q,-]_G + d_{\mathscr{H}}) \underset{\Psi_\natural^\bullet}{\overset{\Phi_\natural^\bullet}{\rightleftarrows}} (\Omega_X^{0,\bullet}(\mathcal{D}_{\mathrm{poly}}^\bullet(X)), \bar{\partial} + d_{\mathscr{H}}).$$

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The construction of this contraction needs the following two results:

Proposition (Liao-Stiénon)

Any affine connection ∇ on $\mathcal{M}$ determines an isomorphism of coalgebras

$$\mathrm{pbw} = \mathrm{pbw}^\nabla : \Gamma(ST_\mathcal{M}) \to \mathcal{D}(\mathcal{M}).$$

Proposition (Laurent-Gengoux-Stiénon-Xu)

For each $(1,0)$-connection $\bar{\nabla}$ on $T_X^{1,0}$, there is an isomorphism of coalgebras

$$\overline{\mathrm{pbw}} = \mathrm{pbw}^{\bar{\nabla}} : \Gamma(ST_X^{1,0}) \to \mathcal{U}(T_X^{1,0}),$$

which generalizes exp *from the holomorphic exponential map when X is Kähler.*

A coalgebra contraction for the space of differential operators

Recall that the symmetric tensor product of the contraction for $\Gamma(T_{\mathcal{M}})$ over $\Omega^{0,\bullet}_X(T^{1,0}_X)$ gives the following $\Omega^{0,\bullet}_X$-coalgebra contraction

$$H \circlearrowright (\Gamma(ST_{\mathcal{M}}), L_Q) \underset{\Psi^\bullet}{\overset{\Phi^\bullet}{\rightleftarrows}} (\Omega^{0,\bullet}_X(ST^{1,0}_X), \bar{\partial}).$$

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The formal exponential map pbw induces a new differential on $\Gamma(ST_{\mathcal{M}})$

$$D = \mathrm{pbw}^{-1} \circ [Q, -]_G \circ \mathrm{pbw} = L_Q + \Theta,$$

where Θ, being a perturbation of L_Q, is a coderivation on $\Gamma(ST_{\mathcal{M}})$. Using perturbation lemma, we get a new coalgebra contraction

$$H_\flat \circlearrowright (\Gamma(ST_{\mathcal{M}}), D) \underset{\Psi_\flat}{\overset{\Phi_\flat}{\rightleftarrows}} (\Omega_X^{0,\bullet}(ST_X^{1,0}), \bar{D} = \bar{\partial} + \overline{\Theta}),$$

where $\bar{D} = \overline{\mathrm{pbw}}^{-1} \circ \bar{\partial} \circ \overline{\mathrm{pbw}}$ is the exactly the differential induced from the isomorphism $\overline{\mathrm{pbw}}$.

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where $\bar{D} = \overline{\mathrm{pbw}}^{-1} \circ \bar{\partial} \circ \overline{\mathrm{pbw}}$ is the exactly the differential induced from the isomorphism $\overline{\mathrm{pbw}}$. Hence, we get an coalgebra contraction on the differential operators

$$H_\natural \circlearrowright (\mathcal{D}(\mathcal{M}), [Q, -]_G) \underset{\Psi_\natural}{\overset{\Phi_\natural}{\rightleftarrows}} (\Omega_X^{0,\bullet}(\mathcal{D}_{\mathrm{poly}}^0(X)), \bar{\partial}).$$

Isomorphism of Gerstenhaber algebras on the cohomology level

Applying the tensor trick of contractions and the perturbation lemma, we obtain the desired $\Omega_X^{0,\bullet}$-coalgebra contraction

$$\breve{H}_\natural \circlearrowright (\mathrm{tot}_\oplus^\bullet(\mathcal{D}_{\mathrm{poly}}(\mathcal{M})), \mathcal{Q} + d_{\mathscr{H}}) \underset{\Psi_\natural^\bullet}{\overset{\Phi_\natural^\bullet}{\rightleftarrows}} (\Omega_X^{0,\bullet}(\mathcal{D}_{\mathrm{poly}}^\bullet(X)), \bar{\partial} + d_{\mathscr{H}}).$$

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Passing to the cohomology level, we have

Proposition (Chen-Xiang-Xu)

There is an isomorphism of Gerstenhaber algebras

$$\Phi_\natural^\bullet : H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[-1], \mathcal{Q} + d_{\mathscr{H}}) \xrightarrow{\cong} H^\bullet(\Omega_X^{0,\bullet}(\mathcal{D}_{\mathrm{poly}}^\bullet(X)), \bar{\partial} + d_{\mathscr{H}}).$$

As a consequence, we have an isomorphism of Gerstenhaber algebras

$$\Phi_\natural^\bullet : H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[-1], \mathcal{Q} + d_{\mathscr{H}}) \cong HH^\bullet(X).$$

Application

Note that $\Phi_\natural\,|_{D^{\leq 1}(\mathcal{M})} = \Phi$. By the definitions of hkr on dg manifolds and on the complex manifold, we have the following

$$\begin{array}{ccc}
\bigoplus_{p,q}(\mathcal{T}^p_{\text{poly}}(\mathcal{M}))^q & \xrightarrow{\text{hkr}} & \bigoplus_{p,q}(\mathcal{D}^p_{\text{poly}}(\mathcal{M}))^q \\
\Big\downarrow{\scriptstyle \Phi^{p+1,0}} & & \Big\downarrow{\scriptstyle \Phi^{p+1}_\natural} \\
\bigoplus_{p,q}\Omega^{0,q}_X(\wedge^{p+1}T_X) & \xrightarrow{\text{hkr}} & \bigoplus_{p,q}\Omega^{0,q}_X(\mathcal{D}^p_{\text{poly}}(X)).
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Combining with the commutative diagram on the compatibility between projection $\Phi^{\bullet,0}$ and contraction with square root of Todd classes, we have the following commutative diagram in the poster

$$\begin{array}{ccc} H^\bullet(\mathrm{tot}_\oplus(\mathcal{T}_{\mathrm{poly}}(\mathcal{M}))[-1], \mathcal{Q}) & \xrightarrow[\cong]{\mathrm{hkr}\circ \mathrm{Td}^{1/2}_{(\mathcal{M},Q)}} & H^\bullet(\mathrm{tot}_\oplus(\mathcal{D}_{\mathrm{poly}}(\mathcal{M}))[-1], \mathcal{Q} + d_{\mathscr{H}}) \\ \Phi^{\bullet,0}\big\downarrow\cong & & \cong\big\downarrow\Phi_\natural^\bullet \\ H^\bullet(X, \wedge^\bullet T_X) & \xrightarrow[\cong]{\mathrm{hkr}\circ \mathrm{Td}^{1/2}_{T^{\mathbb{C}}X/T^{0,1}_X}} & HH^\bullet(X). \end{array}$$

End

Thank you.