WORKSHOP AND SCHOOL ON GEOMETRIC ANALYSIS AND DISCRETE GEOMETRY

Date: February 10–13, 2020  Place: Room 8101, KIAS

1. SCHOOL ON DISCRETE GEOMETRY

Discrete Omega surfaces

Mason Pember, Politecnico di Torino
mason.j.w.pember@bath.edu

Omega surfaces are a large class of surfaces that constitute an integrable system. Discovered by Demoulin in 1911, they are characterised by the existence of an enveloping isothermic sphere congruence. Examples include linear Weingarten surfaces and isothermic surfaces. In this talk, I shall present the smooth and discrete theories of these surfaces, utilising Laguerre geometry.

Stability problem of equilibrium discrete curves and surfaces modelled by anisotropic energy

Yoshiki Jikumaru, Kyushu University
y-jikumaru@math.kyushu-u.ac.jp

An equilibrium surface for anisotropic energy gives a mathematical model of soap bubbles and crystals and it is characterized by a constant anisotropic mean curvature (CAMC) surface. In this talk, we formulate its discretization for curves and surfaces from variational viewpoint inspired by the paper by Pothier and Rossman (J. reine angew. Math., 2002), and discuss its stability problem. We show that the regular polygons are the only equilibrium discrete curves for the length functional and show the instability of non-convex regular polygons. We also show the conservation law for discrete CAMC curves and a criterion for unstable discrete CAMC curves, and give a stability result for discrete CAMC surfaces which is a generalization of the result by Polthier and Rossman.

Discrete Weierstrass type representations

Mason Pember, Politecnico di Torino
mason.j.w.pember@bath.edu

The Weierstrass-Enneper representation has been used extensively to create interesting examples of minimal surfaces in Euclidean 3-space from holomorphic functions. It is well known that this mechanism can be interpreted as the Christoffel transform of a map into the unit sphere. Bobenko and Pinkall (1996) used this interpretation to obtain a Weierstrass representation for discrete minimal nets in Euclidean 3-space.
There are many other Weierstrass type representations for surfaces (smooth and discrete) with certain curvature in 3-dimensional space forms. In this talk we shall see how these can be unified using the theory of Omega surfaces.

**Topologies and singularities of discrete developable surfaces**

Kosuke Naokawa, Hiroshima Institute of Technology  
k.naokawa.ec@cc.it-hiroshima.ac.jp

A ruled surface, which is generated by a continuous motion of a line in Euclidean 3-space $\mathbb{R}^3$, is called developable if its Gaussian curvature vanishes identically. Developable surfaces have the property of ruled surfaces locally isometric to $\mathbb{R}^2$ with the standard metric. Planes, cones, cylinders and tangential surfaces are typical examples. This property gives a natural idea for discretizing developable surfaces. In fact, a ‘discrete’ motion of a line, that is, a sequence of lines in $\mathbb{R}^3$ is called a discrete developable surface if any adjacent two lines of the sequence lie in a plane in $\mathbb{R}^3$, as in the following figures:

![Figure 1. Discrete versions of a plane, cylindrical surface, cone and tangential surface, respectively.](image)

In this talk, we give several results related to topologies and ‘singularities’ of discrete developable surfaces. This project is based on a joint work with Chirstian Müller (TU-Wien).

**Curvature for polyhedral surfaces**

Christian Müller, Technische Universität Wien  
cmueller@geometrie.tuwien.ac.at

We explore discrete curvature notions in particular for polyhedral surfaces but also for special polygonal surfaces. We will lay our focus on vanishing and constant discrete and semi-discrete mean curvature notions which discretize minimal and cmc surfaces.

**A hinged linkage mechanism that follows from discrete integrable equations**

Hyeongki Park, Kyushu University  
h-paku@math.kyushu-u.ac.jp

We consider a family of linkage mechanisms called the Kaleidocycles, which consist $n$-copies of a rigid body joined together by hinges to form a ring. We formulate Kaleidocycles
as discrete closed space curves, and present particular paths in the configuration space of
them, which are governed by the semi-discrete mKdV and sine-Gordon equations.

**Discrete channel linear Weingarten surfaces**

Denis Polly, Technische Universität Wien
dpolly@geometrie.tuwien.ac.at

Channel surfaces are characterized by a number of equivalent properties. Recently,
a discretisation preserving these properties has been suggested. We investigate discrete
channel surfaces with a special linear relationship between their discrete principal curva-
tures.

**Discrete Koenigs nets and applications**

Christian Müller, Technische Universität Wien
cmueller@geometrie.tuwien.ac.at

Koenigs nets constitute a special subclass of conjugate nets with their characterizing
properties being invariant under projective transformations. We will discuss structure
preserving discretizations and how they appear in applications.
2. **Workshop on Geometric Analysis**

**Stable anisotropic capillary hypersurfaces in a wedge**

Miyuki Koiso, Kyushu University
koiso@math.kyushu-u.ac.jp

We study surfaces with constant anisotropic mean curvature in the domain bounded by a wedge in the three-dimensional euclidean space, which are critical points of “the anisotropic surface energy and the wetting energy” for variations preserving the enclosed volume. We show a uniqueness result for local minimizers of the total energy. The result is generalized to hypersurfaces in higher dimensional spaces. Moreover, the result is applied to the uniqueness problem for stable solutions of partially-crystalline variational problems for piecewise smooth hypersurfaces.

**Existence of self-similar solution of the Inverse mean curvature flow**

Kin Ming Hui, Academia Sinica
makmhui66@gmail.com

We will give a new proof of a recent result of P. Daskalopoulos, G. Huisken and J. R. King on the existence of self-similar solution of the inverse mean curvature flow which is the graph of a radially symmetric solution in $\mathbb{R}^n$, $n \geq 2$, of the form $u(x,t) = e^{\lambda t} f(e^{-\lambda t}x)$ for any constants $\lambda > \frac{1}{n-1}$ and $\mu < 0$ such that $f(0) = \mu$. More precisely we will give a new proof of the existence of a unique radially symmetric solution $f$ of the equation

$$\text{div} \left( \frac{\nabla f}{\sqrt{1 + |\nabla f|^2}} \right) = \frac{1}{\lambda} \cdot \frac{\sqrt{1 + |\nabla f|^2}}{x \cdot \nabla f}$$

in $\mathbb{R}^n$, $f(0) = \mu$, for any $\lambda > \frac{1}{n-1}$ and $\mu < 0$, which satisfies $f_r(r) > 0$, $f_{rr}(r) > 0$ and $rf_r(r) > f(r)$ for all $r > 0$. We will also prove that $\lim_{r \to \infty} \frac{rf_r(r)}{f(r)} = \frac{\lambda(n-1)}{\lambda(n-1)-1}$.

**Relative Kähler-Ricci flow on a holomorphic family of strongly pseudoconvex domains**

Sungmin Yoo, IBS
sungmin@ibs.re.kr

In 2012, Schumacher proved that the variation of Kähler-Einstein metrics on a holomorphic family of canonically polarized compact Kähler manifolds is positive definite on the total space. In his paper, he showed that the geodesic curvature, which measures the positivity of the horizontal direction, satisfies a certain elliptic PDE. Applying the maximum principle to this PDE, he obtained the positivity. In 2013, Berman proved a parabolic version of the Schumacher’s result. More precisely, he proved that the geodesic curvature of a holomorphic family of canonically polarized compact Kähler manifolds satisfies a parabolic equation. A parabolic maximum principle implies that the positivity of the geodesic curvature is preserved along the Kähler-Ricci flow. In this talk, we will briefly introduce the results of Schumacher and Berman and show how to apply Berman’s method to a holomorphic family of strongly pseudoconvex domains, which are noncompact complete Kähler manifolds. This is joint work with Young-Jun Choi.
Long-Time Solutions to the Kähler-Ricci Flow
Frederick Tsz-Ho Fong, Hong Kong University of Science and Technology
frederick.fong@ust.hk

The speaker will discuss the local estimates of the Kähler-Ricci flow on compact Kähler manifolds with semi-ample canonical line bundles. On such a manifold, the Kähler-Ricci flow has long-time solutions and its convergence and singular behaviors have been widely studied by various authors. In this talk, the speaker will discuss his works on this topic, in particular showing that the set of fibers (either singular or regular) on which the Riemann curvature blow up along the flow is an invariant set independent of the choice of initial Kähler metric. The research conducted is partially supported by Hong Kong RGC Grants #26301316 and #16300018.

Properties of the solitons for the inverse mean curvature flow
Daehwan Kim, KIAS
daehwan@kias.re.kr

The inverse mean curvature flow has been extensively studied not only as a type of geometric flows, but also for its applications to geometric inequalities. Analyzing special solutions of geometric flow is a natural way to understand, and so does the inverse mean curvature flow. In this talk, we focus on the homothetic and translating solitons for the inverse mean curvature flow that are self-similar solutions deformed by only homothety and translation under the flow, respectively. To be specific, we introduce several examples of the solitons and the incompleteness for the solitons are observed from several examples and then, the incompleteness of any translating soliton and the homothetic solitons with restrict homothetic ratio, namely, $C < \frac{1}{n}$, can be proved by applying maximum principle. Their area growths are obtained.

The second Yamabe invariant
Jinwoo Shin, KIAS
shinjin@kias.re.kr

The Yamabe invariant is a crucial factor in solving the Yamabe problem. In this talk, we introduce the generalization of the Yamabe invariant. We define the second Yamabe invariant as the infimum of the second eigenvalue of the Yamabe operator over the metrics conformal to $g$ and of volume 1. We discuss when it is attained and its properties.

Contracting convex surfaces by mean curvature flow with free boundary on convex barriers
Martin Man-chun Li, Chinese University of Hong Kong
martinli@math.cuhk.edu.hk

We consider the mean curvature flow of compact convex surfaces in Euclidean 3-space with free boundary lying on an arbitrary convex barrier surface with bounded geometry. When the initial surface is sufficiently convex, depending only on the geometry of the
barrier, the flow contracts the surface to a point in finite time. Moreover, the solution is asymptotic to a shrinking half-sphere lying in a half space. This extends, in dimension two, the convergence result of Stahl for umbilic barriers to general convex barriers. We introduce a new perturbation argument to establish fundamental convexity and pinching estimates for the flow. Our result can be compared to a celebrated convergence theorem of Huisken for mean curvature flow of convex hypersurfaces in Riemannian manifolds. This is joint work with Sven Hirsch. (These works are partially supported by RGC grants from the Hong Kong Government.)

**Duality of boundary value problems for minimal and maximal surfaces**

Shintaro Akamine, Nagoya University  
s-akamine@math.nagoya-u.ac.jp

In 1966, Jenkins and Serrin gave existence and uniqueness results for infinite boundary value problems of minimal surfaces in the Euclidean space, and after that such solutions have been studied by using the univalent harmonic mapping theory. In this talk, we show that there exists a one-to-one correspondence between solutions of infinite boundary value problems for minimal surfaces and those of lightlike line boundary problems for maximal surfaces in the Lorentz-Minkowski space. We also investigate some symmetry relations associated with the above correspondence together with their conjugations as below. This talk is based on the preprint arXiv:1909.00975, which is the joint work with Hiroki Fujino (Nagoya University).
Counterexamples to graph preservation under mean curvature flow

Valentina-Mira Wheeler, University of Wollongong
valentina-mira_wheeler@uow.edu.au

Mean curvature flow is the steepest descent gradient flow for the area functional, making it a difficult flow to obtain a long time existence result for geometric objects that flow without an additional restriction. One can see this in a “simple” way by using convex objects as barrier. The celebrated result of Huisken showing that convex bodies under mean curvature flow will shrink out of existence in finite time allows us to prove the extinction of any other object contained inside (or lose regularity before). One property of the solution that facilitates the long time existence proof is that of being graphical. A mean curvature flow solution that is graphical is equivalent to a second order quasilinear strictly parabolic partial differential equation for which one can employ parabolic methods to obtain long time existence. Under graphicality Ecker and Huisken proved in ’89 that entire solutions of mean curvature flow exist for all times. For boundary value problems one has to consider further arguments to manage maintaining initial graphicality up to and including the boundary. In this talk we show that for hypersurfaces moving by mean curvature flow with free boundary, preservation of graphicality holds only in very special circumstances. That is we prove that for any non-cylindrical smooth support hypersurface there exist smooth mean curvature flows with graphical initial data and free boundary on this hypersurface that become non-graphical in finite time. This is joint work with Ben Andrews (ANU).