Induced second-order gravitational waves

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Outline

1. Introduction
2. Equation of motion for GWs
3. 2nd order solutions
   - How to proceed
   - Induced GWs during MD and RD
4. Gauge dependence of induced GWs
5. Conclusions

Why non-linear tensor perturbations a.k.a. GWs?

- Persistent (even during dS)
- Observable signature on small scales
- A new window to the universe

Induced gravitational waves?

Test of GR or any modification and possible observational signatures

Reduction of traceless equation

In ADM formulation, dynamics in the (3D) geometric objects e.g. K_{ij}

Traceless part of evolution of g_{ij} + SVT decomposition

$$g_{ij} = a^2(t)\left[(1+2\varphi)\delta_{ij} + 2\gamma_{,ij} + 2C_{(i,j)} + 2h_{ij}\right]$$

$$\Longrightarrow \quad \ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\Delta}{a^2}h_{ij} + \frac{1}{a^2}\left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\Delta\right)\left(-\varphi + \cdots\right)$$

$$= \text{(non-linear combinations of perturbations)} \equiv s_{ij}$$

Evolution equation for GWs

Decomposition using polarization tensors $e^{\lambda}_{ij}(\boldsymbol{k})$ ($\lambda = +$ and $\times$)

$$h_{ij}(t,\boldsymbol{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \sum_{\lambda=+,\times} h_\lambda(t,\boldsymbol{k}) e^{\lambda}_{ij}(\boldsymbol{k})$$

$$\underbrace{\ddot{h}_\lambda + 3H\dot{h}_\lambda + \frac{k^2}{a^2}h_\lambda}_{\text{linearized in tensor perturbations}} = e^{ij}_{\lambda}(\boldsymbol{k}) s_{ij}(\boldsymbol{k})$$

Non-linear sources

Scalar-vector-tensor components are all mixed

- Scalar-scalar (Mollerach et al. 2004, Ananda et al. 2007, Baumann et al. 2007, Assadullahi et al. 2009...)
  Newtonian gauge, other gauges only very recently
  (Hwang, Jeong and Noh 2017, Tomikawa and Kobayashi 2019, De Luca et al. 2019, Inomata and Terada 2019...)
- Scalar-vector
- Vector-vector
- Vector-tensor
  Interesting in the presence of vector source, e.g. magnetic field

✓ Scalar-tensor

✓ Tensor-tensor

How the sources look like

$$
\begin{aligned}
s_{ij}^{(ss)} =& \frac{1}{a^3}\frac{d}{dt}\left[a\left(2\varphi\chi_{,ij}+\varphi_{,i}\chi_{,j}+\varphi_{,j}\chi_{,i}\right)\right]+\frac{1}{a^2}\left(\kappa\chi_{,ij}-4\varphi\varphi_{,ij}-3\varphi_{,i}\varphi_{,j}\right)+\frac{1}{a^4}\chi^{,k}{}_{,i}\chi_{,jk} \\
&+\frac{1}{a^2}\left[2\alpha\dot{\chi}_{,ij}-H\alpha\chi_{,ij}+\dot{\alpha}\chi_{,ij}-2(\alpha+\varphi)\alpha_{,ij}-\alpha_{,i}\alpha_{,j}-2\varphi_{(,i}\alpha_{,j)}\right]+8\pi G(\rho+p)v_{,i}v_{,j}\,, \qquad (46)
\end{aligned}
$$

$$
\begin{aligned}
s_{ij}^{(tt)} =& \frac{d}{dt}\left(2h_i^k\dot{h}_{jk}\right)+3H\left(2h_i^k\dot{h}_{jk}\right) \\
&+\frac{1}{a^2}\left[2h^{kl}\left(h_{il,jk}+h_{jl,ik}-h_{ij,kl}-h_{kl,ij}\right)-2h_i^k\Delta h_{jk}-h^{kl}{}_{,i}h_{kl,j}+2h_i^{k,l}\left(h_{jl,k}-h_{jk,l}\right)\right], \qquad (47)
\end{aligned}
$$

$$
\begin{aligned}
s_{ij}^{(st)} =& \frac{d}{dt}\left[\dot{h}_{ij}\alpha+2\left(\varphi\dot{h}_{ij}+\dot{\varphi}h_{ij}+\frac{1}{a^2}h_i^k\chi_{,jk}\right)+\frac{\chi^{,k}}{a^2}\left(h_{ik,j}+h_{jk,i}-h_{ij,k}\right)\right] \\
&+3H\left[\dot{h}_{ij}\alpha+2\left(\varphi\dot{h}_{ij}+\dot{\varphi}h_{ij}+\frac{1}{a^2}h_i^k\chi_{,jk}\right)+\frac{\chi^{,k}}{a^2}\left(h_{ik,j}+h_{jk,i}-h_{ij,k}\right)\right] \\
&+\alpha\frac{d}{dt}\left(\dot{h}_{ij}\right)-\frac{1}{a^2}\chi^{,k}\dot{h}_{ij,k}+\kappa\dot{h}_{ij}+\frac{1}{a^2}\left[-2h_i^k\alpha_{,jk}-\left(h_{ik,j}+h_{jk,i}-h_{ij,k}\right)\alpha^{,k}\right]+\frac{1}{a^2}\chi_{,i}{}^{,k}\dot{h}_{jk}-\frac{1}{a^2}\chi_{,j}{}^{,k}\dot{h}_{ik} \\
&+\frac{1}{a^2}\left[2\left(-2\varphi\Delta h_{ij}+h_j^k\varphi_{,ik}-h_{ij}\Delta\varphi\right)+\varphi^{,k}\left(h_{ik,j}+h_{jk,i}-3h_{ij,k}\right)\right]. \qquad (48)
\end{aligned}
$$

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Green's function solution

Let the homogeneous solutions of $Ly(x) = r(x)$ be $y_1(x)$ and $y_2(x)$

$$y(x) = (\text{combination of } y_1 \text{ and } y_2 \text{ according to B.C.}) + \int d\tilde{x} r(\tilde{x}) \underbrace{\frac{y_1(\tilde{x})y_2(x) - y_2(\tilde{x})y_1(x)}{y_1(\tilde{x})y_2'(\tilde{x}) - y_2(\tilde{x})y_1'(\tilde{x})}}_{\equiv G(x,\tilde{x})}$$

Linear solutions for scalar and tensor perturbations are well known

Integral form of induced GWs

If souced by 2 linear perts X and Y, during both MD and RD,

$$h_\lambda(\eta,\boldsymbol{k}) = \frac{1}{a}\int_0^\eta d\tilde{\eta}\left[a^3(\tilde{\eta})e_\lambda^{ij}(\boldsymbol{k})s_{ij}(\boldsymbol{k})\right]G(\eta,\tilde{\eta})$$
$$= \int\frac{d^3q}{(2\pi)^3}\left[e_\lambda^{ij}(\boldsymbol{k})(\cdots)_{ij}\right]X_0(\boldsymbol{k}-\boldsymbol{q})Y_0(\boldsymbol{q})\int_0^x d\tilde{x}K(\tilde{x},\boldsymbol{k},\boldsymbol{q})$$

1. $e_\lambda^{ij}(\boldsymbol{k})(\cdots)_{ij}$: Projection of $e_\lambda^{ij}(\boldsymbol{k})$
2. $X_0(\boldsymbol{k})$ and $Y_0(\boldsymbol{k})$: Initial amp of X and Y, i.e. $\mathscr{R}(\boldsymbol{k})$ and/or $h_0^\lambda(\boldsymbol{k})$
3. $\int_0^x d\tilde{x}K$: Kernel as a function of time and momenta

Scalar-scalar induced GWs

In the Newtonian gauge

$$h_\lambda(\eta,\boldsymbol{k}) = \frac{6}{5}\int\frac{d^3q}{(2\pi)^3}\left[e_\lambda^{ij}(\boldsymbol{k})q_iq_j\right]\mathscr{R}(\boldsymbol{k}-\boldsymbol{q})\mathscr{R}(\boldsymbol{q})\left[1-3\frac{j_1(k\eta)}{k\eta}\qquad\qquad\right]$$

Scalar-scalar induced GWs

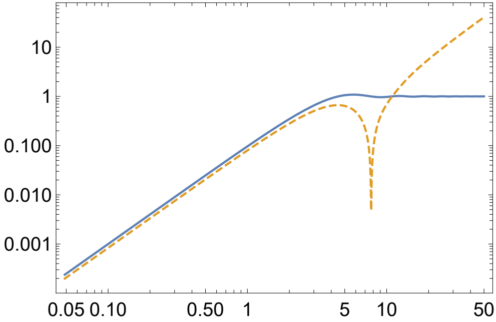
In the Newtonian gauge and comoving gauge

$$h_\lambda(\eta,\boldsymbol{k}) = \frac{6}{5}\int\frac{d^3q}{(2\pi)^3}\left[e_\lambda^{ij}(\boldsymbol{k})q_iq_j\right]\mathscr{R}(\boldsymbol{k}-\boldsymbol{q})\mathscr{R}(\boldsymbol{q})\left[1-3\frac{j_1(k\eta)}{k\eta}-\frac{(k\eta)^2}{60}\right]$$

Scalar-scalar induced GWs

In the Newtonian gauge and comoving gauge

$$h_\lambda(\eta,\boldsymbol{k}) = \frac{6}{5}\int\frac{d^3q}{(2\pi)^3}\left[e^{ij}_\lambda(\boldsymbol{k})q_iq_j\right]\mathscr{R}(\boldsymbol{k}-\boldsymbol{q})\mathscr{R}(\boldsymbol{q})\left[1-3\frac{j_1(k\eta)}{k\eta}-\frac{(k\eta)^2}{60}\right]$$



Scalar-tensor induced GWs

Scalar-tensor case can be analytically soluble

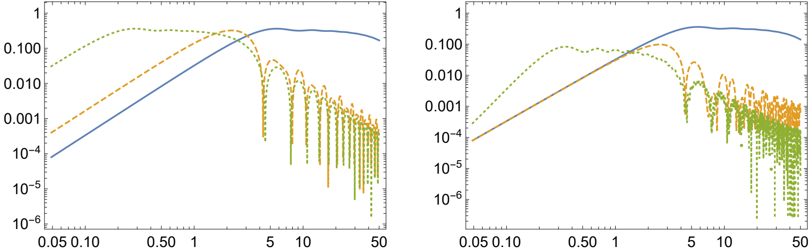
- Newtonian gauge

$$h_\lambda(\eta,\boldsymbol{k}) = \frac{18}{5}\int\frac{d^3q}{(2\pi)^3}\left[e_\lambda^{ij}(\boldsymbol{k})e_{ij}^{\lambda'}(\boldsymbol{q})\right]\mathscr{R}(\boldsymbol{k}-\boldsymbol{q})h_0^{\lambda'}(\boldsymbol{q})\left[\frac{j_1(q\eta)}{q\eta}+\frac{j_1(k\eta)}{k\eta}\right]\frac{q^2+k^2}{q^2-k^2}$$

- Comoving gauge

$$h_\lambda(\eta,\boldsymbol{k}) = 6\int\frac{d^3q}{(2\pi)^3}\left[e_\lambda^{ij}(\boldsymbol{k})e_{ij}^{\lambda'}(\boldsymbol{q})\right]\mathscr{R}(\boldsymbol{k}-\boldsymbol{q})h_0^{\lambda'}(\boldsymbol{q})$$
$$\times\left[\frac{q^2+5k^2}{5(q^2-k^2)}\frac{j_1(k\eta)}{k\eta}-\frac{5q^2+7k^2}{10(q^2-k^2)}\frac{j_1(q\eta)}{q\eta}+\frac{1}{10}j_0(q\eta)\right]$$

Kernels for scalar-tensor induced GWs

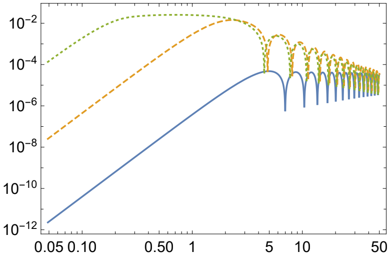


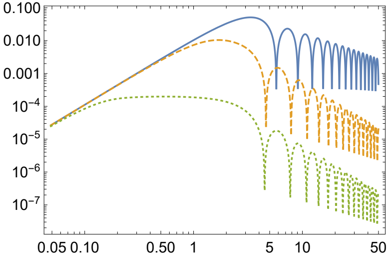
Comoving gauge kernel exhibits more rapid oscillations

Solutions with tensor-tensor sources

Tensor-tensor source is also analytically soluble

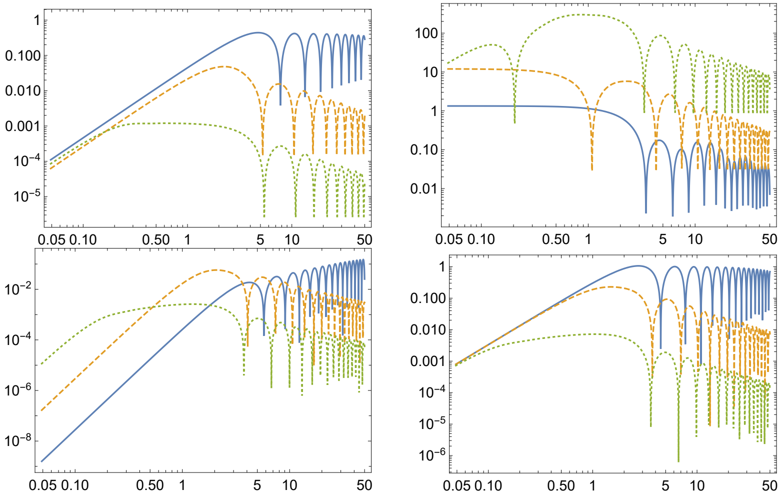
$$h_\lambda(\eta, \boldsymbol{k}) = \int \frac{d^3q}{(2\pi)^3}\left[(\cdots)F_{\rm MD}(\eta, \boldsymbol{k}, \boldsymbol{q}) + (\cdots)G_{\rm MD}(\eta, \boldsymbol{k}, \boldsymbol{q})\right]$$





SS, ST and TT induced GWs during RD

Scalar perts in the Newtonian gauge and we can proceed similarly

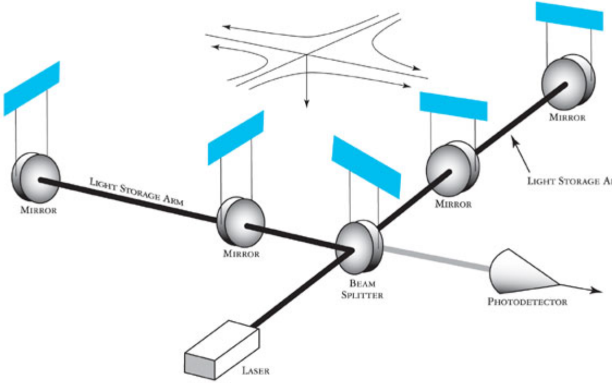


Gauge dependence of induced GWs

Induced (by scalar perts) GWs are **gauge dependent** and thus are not (direct) observable

But GWs ARE observable

What we interpret as GWs in experiments



We interpret the **local anisotropic tidal field** as GWs

Global and local coordinate systems

GW observatories are fixed (= free-falling) on Earth (= weak field approx & local Minkowski space-time)

$$\overbrace{\tilde{g}_{\rho\sigma}(\tilde{x})}^{\text{local Minkowski}} = \underbrace{\frac{\partial x^\mu}{\partial \tilde{x}^\rho}\frac{\partial x^\nu}{\partial \tilde{x}^\sigma}}_{\text{global } (x^\mu) \text{ to local } (\tilde{x}^\mu) \text{ transformation}} \overbrace{g_{\mu\nu}(x)}^{\text{global FRW}}$$

We find deviations away from our free-falling observer location

$$ds^2 = -\Big(1 + \underbrace{\widetilde{R}_{0i0j}\tilde{x}^i\tilde{x}^j}_{\equiv 2\widetilde{\Phi}, \text{ local gravitational potential}}\Big)d\tilde{t}^2 + \cdots$$

Local GWs from global scalar perturbations

Local gravitational potential $\widetilde{\Phi} = \widetilde{R}_{0i0j}\tilde{x}^i\tilde{x}^j/2$ with

$$\widetilde{R}_{0i0j} = \left[\left(e_0^0\right)^2 e_i^p e_j^q - 2e_0^p e_i^0 e_0^0 e_j^q\right] R_{0p0q} + 2e_0^p e_i^q e_0^0 e_j^r R_{pq0r} + e_0^p e_i^q e_0^r e_j^s R_{pqrs}$$
$$= \text{complicated 2nd-order expression with global perturbations}$$

We can read the anisotropic tidal component

Including only global GWs $g_{ij} = a^2(\delta_{ij} + h_{ij})$ gives what we expect

$$\frac{d^2\tilde{x}^i}{d\tilde{t}^2} = -\frac{\partial\widetilde{\Phi}}{\partial\tilde{x}^i} = \left(\dot{H} + H^2\right)\tilde{x}^i + \frac{1}{2}\Big(\underbrace{\ddot{h}^i{}_j + 2H\dot{h}^i{}_j}_{=\frac{d^2\tilde{h}_{ij}^{\rm TT}}{d\tilde{t}^2}}\Big)\tilde{x}^j$$

Conclusions

1. Pure tensor perturbations are of physical interest
2. Non-linear evolution allows classical feedback
    1. Sourced by all types of linear perturbations
    2. Could be relevant on smaller scales
    3. Accordance or conflict with PBH formation?
    4. “Correct” observable signal?