Combinatorial representation theory arising from quantum symmetric pairs

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KIAS Workshop on combinatorial problems of algebraic origin

Representation theory

= study of symmetries

Symmetries

- arise from various fields of science
- are controlled by groups, rings, algebras, ...
- are studied from many different viewpoints

  e.g. algebraic
  geometric
  analytic
  combinatrial
  physical
  ⋮

Combinatorial problems

$\mathcal{B}_4 := \{ \boxed{1}, \boxed{2}, \boxed{3} \}$: a set

$B_4 := \mathbb{R}\mathcal{B}_4 = \mathbb{R}\boxed{1} \oplus \mathbb{R}\boxed{2} \oplus \mathbb{R}\boxed{3}$: 3-dim'l $\mathbb{R}$-vec. sp.

For $d \in \mathbb{Z}_{>0}$,

$B_4^{\otimes d}$ has a basis $\mathcal{B}_4^{\otimes d} := \{ \boxed{i_1} \otimes \cdots \otimes \boxed{i_d} \mid 1 \leq i_j \leq 3 \ \forall j \}$

Define $wt : \mathcal{B}_4^{\otimes d} \longrightarrow \mathbb{Z}$ by

$$wt(\boxed{i_1} \otimes \cdots \otimes \boxed{i_d}) = (\lambda_1, \lambda_2, \lambda_3)$$

$$\lambda_i := \#\{ j \mid i_j = i \}$$

For $i = 1, 2$, define two linear maps

$$\tilde{E}_i, \tilde{F}_i : B_4^{\otimes d} \longrightarrow B_4^{\otimes d}$$

by induction on d:

<u>$d = 1$</u>

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3}$$

$$b \xrightarrow{i} b' \quad :\Longleftrightarrow \quad \tilde{F}_i(b) = b' \;\&\; \tilde{E}_i(b') = b$$

$$b \not\xrightarrow{i} \quad :\Longleftrightarrow \quad \tilde{F}_i(b) = 0$$

$$\not\xrightarrow{i} b' \quad :\Longleftrightarrow \quad \tilde{E}_i(b') = 0$$

$\underline{d \geq 2}$

$$\begin{tikzcd}
 & \boxed{i} \arrow[r] & \boxed{i+1} & \boxed{j} \\
b \arrow[d] & \bullet \arrow[r] & \bullet \arrow[d] & \bullet \arrow[d] \\
\tilde{F}_i(b) \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] \\
\tilde{F}_i^2(b) \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] & \bullet \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
\tilde{F}_i^{\ell}(b) & \bullet & \bullet & \bullet
\end{tikzcd}$$

$j \neq i, i+1$

$\longrightarrow \; := \; \xrightarrow{i}$

We call $(\mathrm{wt}, \tilde{E}_i, \tilde{F}_i, i = 1, 2)$

a crystal structure on $B_{\square}^{\otimes d}$

Ex. (crystal structure on $B_{\square}^{\otimes 2}$)

$$\begin{tikzcd}
 & \boxed{1} \arrow[r, "1"] & \boxed{2} \arrow[r, "2"] & \boxed{3} \\
\boxed{1} \arrow[d, "1"] & \cdot \arrow[r, "1"] & \cdot \arrow[r, "2"] \arrow[d, "1"] & \cdot \arrow[d, "1"] \\
\boxed{2} \arrow[d, "2"] & \cdot \arrow[d, "2"] & \cdot \arrow[r, "2"] & \cdot \arrow[d, "2"] \\
\boxed{3} & \cdot \arrow[r, "1"] & \cdot & \cdot
\end{tikzcd}$$

$\longrightarrow \; := \; \xrightarrow{1}$

$\longrightarrow \; := \; \xrightarrow{2}$

$\multimap \; B_{\square}^{\otimes 2} = B_1 \oplus B_2$

6-dim   3-dim.

Robinson-Schensted-Knuth correspondence

$$\mathcal{B}_{\square}^{\otimes} \xrightarrow{1:1} \coprod_{\lambda \vdash d} \left(SST_3(\lambda) \times ST_d(\lambda) \right) \ ; \ b \mapsto (P(b), Q(b))$$

$SST_3(\lambda)$ $\parallel$ the set of semistandard tableaux of shape λ in letters $\{1,2,3\}$

Ex. ($b = \boxed{1} \otimes \boxed{2} \otimes \boxed{2} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{2} \otimes \boxed{1}$)

$$\begin{array}{|c|}\hline 1\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline 1&2&2\\\hline\end{array} \rightsquigarrow \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2\\\cline{1-1}\end{array}$$

$$\rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline 2\\\cline{1-1}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1&1&1&3\\\hline 2&2\\\cline{1-2}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2&3\\\cline{1-3}\end{array} \rightsquigarrow \begin{array}{|c|c|c|c|}\hline 1&1&1&1\\\hline 2&2&2\\\cline{1-3} 3\\\cline{1-1}\end{array}$$

$\parallel$ $P(b)$

$$Q(b) = \begin{array}{|c|c|c|c|}\hline 1&2&3&5\\\hline 4&6&7\\\cline{1-3} 8\\\cline{1-1}\end{array}$$

By RSK correspondence,

$$B_4^{\otimes d} = \bigoplus_{\substack{\lambda \vdash d \\ Q \in ST_d(\lambda)}} B[Q] \quad (\text{as vector spaces})$$

$$B[Q] := \mathrm{Span}_{\mathbb{R}} \{ b \in B_4^{\otimes d} \mid Q(b) = Q \}$$

Fact

- $B[Q]$ is a subcrystal of $B_4^{\otimes d}$,

  i.e., it is closed under $\tilde{E}_i, \tilde{F}_i$ $(i = 1, 2)$

- $\exists! \, b \in B[Q]$ s.t. $\tilde{E}_1(b) = \tilde{E}_2(b) = 0$

  $\rightarrow$ $P(b) = \begin{array}{|c|c|c|} \hline 1 & \cdots\cdots & 1 \\ \hline 2 & \cdots & 2 \\ \hline 3 & \cdots & 3 \\ \hline \end{array}$, $\mathrm{wt}(b) = (\lambda_1, \lambda_2, \lambda_3)$

Another structure on $B_4^{\otimes d}$

$\boxed{\pm} := \boxed{1} + \boxed{2}$, $\boxed{0} := \boxed{3}$, $\boxed{-} := \boxed{2} - \boxed{1} \in B_4$

$wt'(\boxed{\pm}) := 2$, $wt'(\boxed{0}) := 0$, $wt'(\boxed{-}) := -2$

$\tilde{A}, \tilde{B} : B_4 \to B_4$: linear map

$$\boxed{\pm} \to \boxed{0} \to \boxed{-}$$

$$b \to b' :\iff \tilde{B}(b) = b' \ \& \ \tilde{A}(b') = b$$

Define wt', $\tilde{A}, \tilde{B}$ on $B_4^{\otimes d}$ by induction on d:

For $\varepsilon_1, \cdots, \varepsilon_{d-1} \in \{+, 0, -\}$,

$$\boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}}\boxed{+} := \begin{cases} \boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}} \otimes \boxed{2} & \text{if } \mathrm{wt}'\left(\boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}}\right) > 0 \\ \boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}} \otimes \left(\boxed{1} + \boxed{2}\right) & = \\ \boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}} \otimes \boxed{1} & < \end{cases}$$

$$\boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}}\boxed{0} := \boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}} \otimes \boxed{3}$$

$$\boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}}\boxed{-} := \begin{cases} \boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}} \otimes \boxed{1} & \text{if } \mathrm{wt}'\left(\boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}}\right) > 0 \\ \boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}} \otimes \left(\boxed{2} - \boxed{1}\right) & = \\ \boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_{d-1}} \otimes \boxed{2} & < \end{cases}$$

$$\mathrm{wt}'\left(\boxed{\varepsilon_1}\boxed{\cdots}\boxed{\varepsilon_d}\right) := \sum_{i=1}^{d} \mathrm{wt}'\left(\boxed{\varepsilon_i}\right)$$

Let $b \in B_4^{\otimes d-1}$ be s.t. $\mathrm{wt}'(b) = a \geq 0$, $\tilde{A}(b) = 0$

b is a linear combination of $\boxed{\varepsilon_1}\cdots\boxed{\varepsilon_d}$'s of wt' a

Then,

$$\tilde{B}^n(b\,\boxed{\pm}) := \begin{cases} b\,\boxed{\pm} & \text{if } n=0 \\ \tilde{B}^{n-1}(b)\,\boxed{0} & 1 \leq n \leq a+1 \\ \tilde{B}^{a}(b)\,\boxed{-} & n = a+2 \\ 0 & n > a+2 \end{cases}$$

$$\tilde{B}^n(\tilde{B}(b)\,\boxed{\pm}) := \begin{cases} \tilde{B}^{n+1}(b)\,\boxed{\pm} & \text{if } 0 \leq n < \frac{a}{2} \\ \tilde{B}^{\frac{a}{2}+1}(b)\,\boxed{\pm} + \tilde{B}^{\frac{a}{2}-1}(b)\,\boxed{-} & n = \frac{a}{2} \\ -\tilde{B}^{n-1}(b)\,\boxed{-} & \frac{a}{2} < n \leq a \\ 0 & n > a \end{cases}$$

$$\tilde{B}^n(b \boxed{-}) := \begin{cases} \tilde{B}^n(b)\boxed{-} & \text{if } 0 \le n < \frac{a}{2}-1 \\ \tilde{B}^{\frac{a}{2}-1}(b)\boxed{-} - \tilde{B}^{\frac{a}{2}+1}(b)\boxed{+} & n = \frac{a}{2}-1 \\ -\tilde{B}^{n+2}(b)\boxed{+} & \frac{a}{2}-1 < n \le a-2 \\ 0 & n > a-2 \end{cases}$$

We call $(\mathrm{wt}', \tilde{A}, \tilde{B})$ an ιcrystal structure on $B_4^{\otimes d}$

Rem.

If $\mathrm{wt}'(b) = a \ge 0$ and $\tilde{A}(b) = 0$, then

$$\tilde{B}^n(b) \ne 0 \iff 0 \le n \le a$$

Ex.

B_4 $\qquad \boxed{+} \longrightarrow \boxed{0} \longrightarrow \boxed{-}$

$B_4^{\otimes 2}$ $\qquad \boxed{+}\boxed{+} \longrightarrow \boxed{+}\boxed{0} \longrightarrow \boxed{0}\boxed{0} \longrightarrow \boxed{-}\boxed{0} \longrightarrow \boxed{-}\boxed{-}$

$\boxed{0}\boxed{+} \longrightarrow \boxed{-}\boxed{+} + \boxed{+}\boxed{-} \longrightarrow -\boxed{0}\boxed{-}$

$\boxed{+}\boxed{-} - \boxed{-}\boxed{+}$

$\rightarrow \quad B_4^{\otimes 2} = B_1 \oplus B_2 \oplus B_3$

$\uparrow$ 5-dim. $\qquad \uparrow$ 3-dim. $\qquad \uparrow$ 1-dim

Problem

(1) Describe $\tilde{A}, \tilde{B} : B_4^{\otimes d} \to B_4^{\otimes d}$

in terms of $\tilde{E}_i, \tilde{F}_i$ $(i=1,2)$

(2) For each $a \in \mathbb{Z}$, set $B_a := \{ b \in B_4^{\otimes d} \mid wt'(b) = a \}$.

Find $\operatorname{Ker} \tilde{A} \cap B_a$

(3) For $a \in \mathbb{Z}$, $\lambda \vdash d$, $Q \in ST_d(\lambda)$,

find $\operatorname{Ker} \tilde{A} \cap B_a \cap B[Q]$

Background

Representation theory of Lie algebras

$$\mathfrak{g} := \mathfrak{gl}_3 = \text{Lie alg. consisting of } (3\times 3) \text{ matrices over } \mathbb{C}$$

$$\cup$$

$$\mathfrak{k} := \{ X = (x_{ij}) \in \mathfrak{g} \mid x_{ij} = (-1)^{i+j-1} x_{ji} \}$$

$$\wr\wr$$

$\mathfrak{so}_3$: the special orthogonal Lie algebra

$$\begin{tikzcd} \mathfrak{g} \arrow[r] & V_q := \mathbb{C}^3 \\ \mathfrak{k} \arrow[u, phantom, "\cup"] \arrow[ur] & \end{tikzcd}$$

: the natural representation of $\mathfrak{g}$, $\mathfrak{k}$

$$\begin{tikzcd} \mathfrak{g} \arrow[r, bend left, phantom, "\curvearrowright"] & V_q^{\otimes d} \\ \cup & \\ \mathfrak{k} \arrow[ru, bend left] & \end{tikzcd}$$

$$X \longmapsto \sum_{i=1}^{d} 1^{\otimes i-1} \otimes X \otimes 1^{\otimes d-i}$$

$$= X \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes X \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes X$$

<u>Note</u>

$\mathcal{B}_q^{\otimes d} \longleftrightarrow$ a basis of $V_q^{\otimes d}$

$B_q^{\otimes d} \longleftrightarrow V_q^{\otimes d}$

$\tilde{E}_i, \tilde{F}_i \longleftrightarrow$ action of E_i, F_i

$\tilde{A}, \tilde{B} \longleftrightarrow$ action of A, B

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Fundamental problems in rep. theory

① Classify the irreducible representations

no subrep's

② Describe the structure of irreducible rep's in a uniform way

③ Decompose a given interesting representation into the sum of irreducible subrep's

Answer to ①

$$\{\text{finite-dim'l irr. rep}\}/\text{isom} \overset{1:1}{\longleftrightarrow} \{\text{dominant weight}\}$$

$$V \longmapsto \text{highest weight of } V$$

$$V(\lambda) \longleftarrow\!\shortmid \ \lambda$$

the irr. rep. of highest weight λ

Ex

dominant weight of $\mathfrak{g} = \mathfrak{gl}_3$ $= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}_{\geq 0}^3 \mid \lambda_1 \geq \lambda_2 \geq \lambda_3\}$

$= \{\text{partition of length } \leq 3\}$

dominant weight of $\mathfrak{k} \simeq \mathfrak{so}_3$ $= \mathbb{Z}_{\geq 0}$

Answer to ② (ongoing from various viewpoints)

- Lusztig's canonical basis (= Kashiwara's global crystal basis)

  gives a basis to each $V(\lambda)$ in a uniform way

  has many favorable properties

  requires knowledge of Lie algebras, algebraic geom.

- Kashiwara's (local) crystal basis

  gives a combinatorial model for $V(\lambda)$

  has many favorable properties

  requires knowledge of combinatorics

Let V be a representation of $\mathfrak{g}$.

A crystal basis is a set $\mathcal{B}$ with a combinatorial structure $(\mathrm{wt}, \tilde{E}_i, \tilde{F}_i)$ parametrising a basis of V.

$\tilde{E}_i, \tilde{F}_i : \mathcal{B} \to \mathcal{B} \amalg \{0\}$

can be extended to linear endomorphisms on $B := \mathbb{R}\mathcal{B}$

Answer to ③ (depends on the given rep.)

Suppose : · V is a representation of $\mathfrak{g}$ with a crystal basis $\mathcal{B}$.

· $V = \bigoplus_{\lambda} V(\lambda)^{\oplus m_\lambda}$: irreducible decomposition

$m_\lambda \in \mathbb{Z}_{\geq 0}$: the multiplicity of $V(\lambda)$ in V.

$$\Rightarrow \ m_\lambda = \# \{ b \in \mathcal{B} \mid \tilde{E}_i(b) = 0 \ \forall i \ \ \& \ wt(b) = \lambda \}$$

Problem (branching law)

Let V be a representation of $\mathfrak{g}$.

Regard V as a rep. of $\mathfrak{k}$

$\longrightarrow$ $V = \bigoplus_a V(a)^{\oplus m_a}$: irr. decomposition as a rep. of $\mathfrak{k}$

Find the multiplicity $m_a \in \mathbb{Z}_{\geq 0}$.

Rem.

Even if V has a crystal basis, crystal basis theory can't tell the answer.

New approach

Let V be a rep. of $\mathfrak{g}$ with a crystal basis $\mathcal{B}$.

$$\mathbf{B} := \mathbb{R}\mathcal{B}$$

rep. theory of ι quantum group (quantum symmetric pair)

new combinatorial structure on $\mathbf{B}$

which models V as a rep. of $\mathfrak{k}$

ι crystal structure

Thm (W.)

Let V be a rep. of $\mathfrak{g}$ with a crystal basis $\mathcal{B}$

Then, $\exists$ an $\imath$crystal structure on $B := R\mathcal{B}$.

Moreover, if we write $V = \bigoplus_a V(a)^{\oplus m_a}$, then

$$m_a = \dim(\operatorname{Ker} \tilde{A} \cap B_a)$$

Summary

combinatorics	representations
$\mathcal{B}_4^{\otimes d}$	canonical basis of $V_4^{\otimes d}$
$B_4^{\otimes d}$	$V_4^{\otimes d}$
$\tilde{E}_i, \tilde{F}_i$	actions of $E_i, F_i \in \mathfrak{g}$
$B_4^{\otimes d} = \bigoplus_Q B[Q]$	irr. decomposition of $V_4^{\otimes d}$
$\tilde{A}, \tilde{B}$	actions of $A, B \in \mathfrak{k}$
$\dim(\operatorname{Ker}\tilde{A} \cap B_a)$	multiplicity of $V(a)$ in $V_4^{\otimes d}$
$\dim(\operatorname{Ker}\tilde{A} \cap B_a \cap B[Q])$	multiplicity of $V(a)$ in $V(\lambda)$ (λ = shape of Q)

Why crystal structure?

- Kashiwara's crystal basis theory
  comes from the rep. theory of quantum groups

$$\mathfrak{g} \underset{q=1}{\overset{q\text{-deform}}{\rightleftarrows}} U_q(\mathfrak{g})$$

$$\text{rep. of } \mathfrak{g} \underset{q=1}{\overset{q\text{-deform}}{\rightleftarrows}} \text{rep. of } U_q(\mathfrak{g})$$

$$\big\downarrow q=\infty$$

$$\text{crystal structure}$$

- $U_q(\mathfrak{k}) \not\subset U_q(\mathfrak{g})$ even though $\mathfrak{k} \subset \mathfrak{g}$

$$
\begin{tikzcd}
\mathfrak{g} \arrow[r, hook] & V \\
\mathfrak{k} \arrow[u, phantom, "\cup"] \arrow[ur, hook, bend left]
\end{tikzcd}
\quad \overset{q\text{-deform}}{\rightsquigarrow} \quad
\begin{tikzcd}
U_q(\mathfrak{g}) \arrow[r, hook] & V_q \\
U_q(\mathfrak{g}) \arrow[u, phantom, "\not\cup"] \arrow[ur, hook, bend left, "\times"] \arrow[r, hook] & V_q'
\end{tikzcd}
$$

- Theory of quantum symmetric pairs provide us a new q-deformation $U^\iota(\mathfrak{k})$ of $\mathfrak{k}$

(quantum grp.

- $U^\iota(\mathfrak{k}) \subset U_q(\mathfrak{g})$

- $$\begin{array}{ccc} \mathfrak{g} & \curvearrowright & V \\ \cup & \nearrow & \\ \mathfrak{k} & & \end{array} \qquad \underset{q\text{-deform}}{\overset{\text{correct}}{\rightsquigarrow}} \qquad \begin{array}{ccc} U_q(\mathfrak{g}) & \curvearrowright & V_q \\ \cup & \nearrow & \\ U^{\imath}(\mathfrak{k}) & & \end{array}$$

$$\Big\downarrow q = \infty$$

$\imath$crystal structure