# $L$-matrices of quiver mutations

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joint work with

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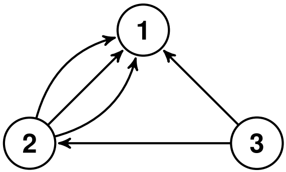
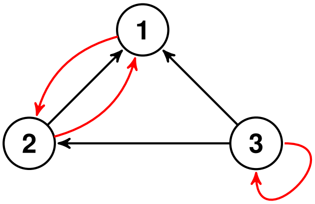
- Let $Q$ be a quiver (= directed graph).
- Assume that $Q$ has no loops or 2-cycles.

Example:




Non-example:



- Quiver mutation at vertex $k$:

  Step 1) For each subquiver

$$\textcircled{i} \longrightarrow \textcircled{k} \longrightarrow \textcircled{j},$$

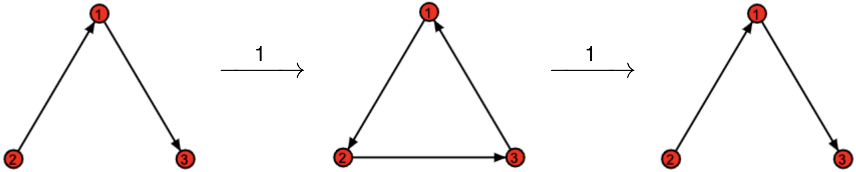
  add a new arrow

$$\textcircled{i} \longrightarrow \textcircled{j}.$$

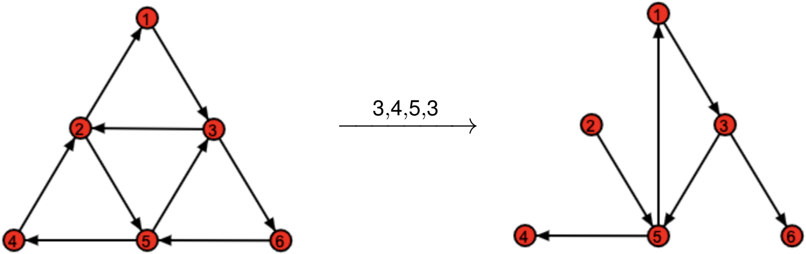
  Step 2) Reverse the directions of any arrows touching $k$.

  Step 3) Remove both arrows of any 2-cycles.

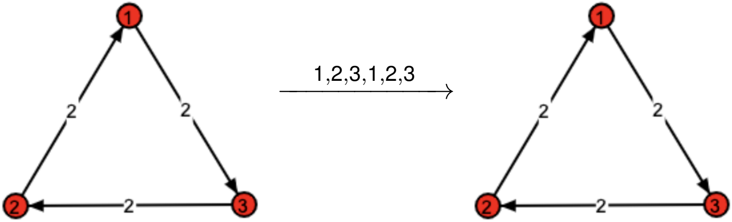
1) Mutation at vertex 1



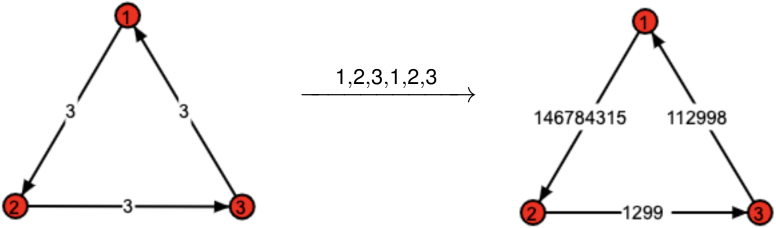

2) Consecutive mutations at vertices 3, 4, 5, 3



3) Consecutive mutations at $1, 2, 3, 1, 2, 3$



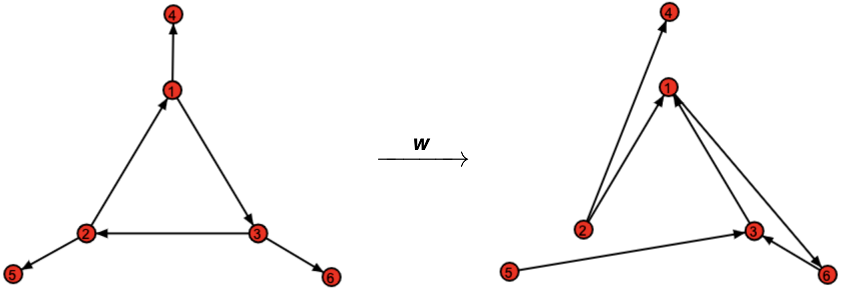

4) Consecutive mutations at $1, 2, 3, 1, 2, 3$



# $c$-vectors and $C$-matrix

- For each vertex $k$ of a quiver $Q$, introduce a new vertex $\tilde{k}$ and an arrow from $k$ to $\tilde{k}$. $\quad k \longrightarrow \tilde{k}$
- The vertices $\tilde{k}$ are frozen in the sense that we do not perform mutations at these vertices.
- After any consecutive mutation sequence $\boldsymbol{w} = [i_1, i_2, \ldots, i_r]$, record the numbers of arrows from $k$ to all the frozen vertices. The resulting row-vectors $c_k^{\boldsymbol{w}}$ are called $c$-vectors.
- The matrix $C^{\boldsymbol{w}} = \begin{bmatrix} c_1^{\boldsymbol{w}} \\ \vdots \\ c_n^{\boldsymbol{w}} \end{bmatrix}$ is called the $C$-matrix.

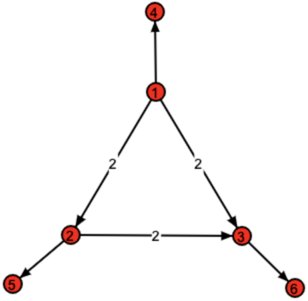
$\boldsymbol{w} = [1, 2, 1, 2, 3]$



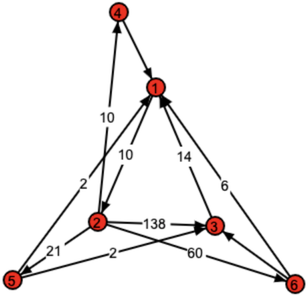

$$c_1^{\boldsymbol{w}} = (0, 0, 1), \quad c_2^{\boldsymbol{w}} = (1, 0, 0), \quad c_3^{\boldsymbol{w}} = (0, -1, -1)$$

$$C^{\boldsymbol{w}} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

$\mathbf{w} = [2, 3, 2, 1]$




$$\xrightarrow{\mathbf{w}}$$




$$C^{\mathbf{w}} = \begin{bmatrix} -1 & -2 & -6 \\ 10 & 21 & 60 \\ 0 & -2 & -1 \end{bmatrix}$$

# Sign coherence of $c$-vectors

**Theorem (DWZ,GHKK)**

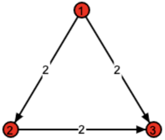
*The $c$-vectors are non-zero, and the entries of a $c$-vector are either all non-negative or all non-positive.*

This theorem is proven by [Derksen–Weyman–Zelevinsky (2010)] and [Gross–Hacking–Keel–Kontsevich (2018)].

- For a nonzero vector $v$, write $v > 0$ if all the entries are non-negative and $v < 0$ if all the entries are non-positive.
- The above theorem says that we have

$$\text{either} \quad c_i^{\boldsymbol{w}} > 0 \quad \text{or} \quad c_i^{\boldsymbol{w}} < 0.$$

A quiver is acyclic if it does not have an oriented cycle. For example,




Assume that $Q$ is acyclic. Then we have an ordering $\prec$ on the set of vertices $\{1, 2, \ldots, n\}$ such that

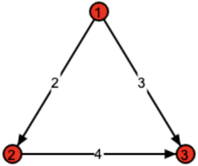
$$i \succ j \quad \Leftrightarrow \quad i \to j.$$

Define a quadratic form

$$q(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{i \succ j} b_{ij} x_i x_j,$$

where $b_{ij}$ is the number of arrows from $i$ to $j$. (Set $b_{ji} = -b_{ij}$.)

For example, when $Q$ is given by




,

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 - 3x_1x_3 - 4x_2x_3.$$

For $\boldsymbol{w} = [2, 1, 3, 2]$, the $c$-vectors are

$$c_1^{\boldsymbol{w}} = (3, 4, 25), \quad c_2^{\boldsymbol{w}} = (-2, -3, 0), \quad c_3^{\boldsymbol{w}} = (0, 0, -1).$$

They satisfy

$$q(3, 4, 25) = 1, \quad q(-2, -3, 0) = 1, \quad q(0, 0, -1) = 1.$$

This is always true with an acyclic quiver.

**Theorem (Kac, 1980; Caldero–Keller, 2006)**

*Assume that Q is acyclic. Then, for any c-vector $c_i^{\mathbf{w}}$, we have*

$$q(c_i^{\mathbf{w}}) = 1.$$

Q: Is this true for non-acyclic quivers?

A: No, but not completely false.

- $\boldsymbol{w} = [2,3,2,1]$

$$\begin{tikzcd} & 2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ 1 \arrow[ur, shift left] \arrow[ur, shift right] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd} \xrightarrow{\boldsymbol{w}} \begin{tikzcd} & 2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ 1 \arrow[ur, shift left] \arrow[ur, shift right] & & 3 \arrow[ll, shift left] \arrow[ll, shift right] \end{tikzcd}$$

$$C^{\boldsymbol{w}} = \begin{bmatrix} -1 & -2 & -2 \\ 2 & 5 & 4 \\ 0 & -2 & -1 \end{bmatrix}$$

- There is no natural ordering for this quiver.
- Fix $1 \succ 2 \succ 3$. Then

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3,$$

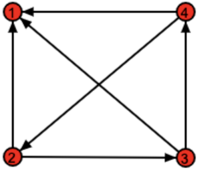
and we have

$$q(-1,-2,-2) = 1, \quad q(2,5,4) = 1, \quad q(0,-2,-1) = 1.$$

It is natural to formulate a conjecture:

> For any quiver $Q$, there exists an ordering $\prec$ such that $c$-vectors are roots of the equation
> $$q(x_1, \ldots, x_n) = 1.$$

However, there is a counterexample.




$$\boldsymbol{w} = [1, 2, 3, 4, 2]$$

The $c$-vector $c_1^{\boldsymbol{w}} = (5, 2, 2, 2)$ is not a root for any ordering.

Idea: When $Q$ is acyclic, $c$-vectors are closely related to reflections.

Let $\alpha_1 = (1, 0, \ldots, 0)$,..., $\alpha_n = (0, \ldots, 0, 1)$ be the standard basis.

Define

$$s_i(\alpha_j) = \begin{cases} \alpha_j + b_{ji}\alpha_i & \text{if } i \prec j, \\ -\alpha_j & \text{if } i = j, \\ \alpha_j - b_{ji}\alpha_i & \text{if } i \succ j. \end{cases}$$

[Caldero–Keller, 2006] says that, when $Q$ is acyclic, the $c$-vectors are real Schur roots, i.e.

$$c_i^{\boldsymbol{w}} = s_{i_1} s_{i_2} \cdots s_{i_r} \alpha_i$$

for "special" sequences $(i_1, i_2, \ldots, i_r)$ of indices.

Since $s_i$'s are isometries, we have, for an acyclic quiver $Q$,

$$q(c_i^{\boldsymbol{w}}) = 1.$$

When $Q$ is non-acyclic, there is a counterexample. Hence, in general,

$$c_i^{\boldsymbol{w}} \neq s_{i_1} s_{i_2} \cdots s_{i_r} \alpha_i.$$

However, the vectors $s_{i_1} s_{i_2} \cdots s_{i_r} \alpha_i$ would still make sense if we can characterize the sequence $(i_1, \dots, i_r)$ for each mutation $\boldsymbol{w}$.

For each $\boldsymbol{w}$, inductively define $r_i^{\boldsymbol{w}}$ with the initial element $s_i$ by

$$r_i^{\boldsymbol{w}[k]} = \begin{cases} r_k^{\boldsymbol{w}} r_i^{\boldsymbol{w}} r_k^{\boldsymbol{w}} & \text{if } b_{ik}^{\boldsymbol{w}} c_k^{\boldsymbol{w}} > 0, \\ r_i^{\boldsymbol{w}} & \text{otherwise,} \end{cases}$$

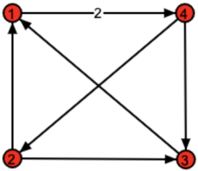
where $b_{ik}^{\boldsymbol{w}}$ are defined by the numbers of arrows in the quiver $Q^{\boldsymbol{w}}$.

Write $r_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}} s_i (g_i^{\boldsymbol{w}})^{-1}$.

## Definition

*Fix an ordering $\prec$. Then the l-vectors $l_i^{\boldsymbol{w}}$ and the L-matrix $L^{\boldsymbol{w}}$ are defined by*

$$l_i^{\boldsymbol{w}} = g_i^{\boldsymbol{w}}(\alpha_i) \quad \textit{and} \quad L^{\boldsymbol{w}} = \begin{bmatrix} l_1^{\boldsymbol{w}} \\ \vdots \\ l_n^{\boldsymbol{w}} \end{bmatrix}.$$




$$\boldsymbol{w} = [3, 4, 1, 3, 4, 3]$$

Take $1 \prec 2 \prec 3 \prec 4$.

$$r_1^{\boldsymbol{w}} = s_3 s_4 s_3 s_1 s_3 s_4 s_3,$$

$$r_2^{\boldsymbol{w}} = s_3 s_4 s_3 s_1 s_3 s_4 s_2 s_4 s_3 s_1 s_3 s_4 s_3,$$

$$r_3^{\boldsymbol{w}} = s_3 s_4 s_1 s_3 s_4 s_3 s_1 s_3 s_1 s_3 s_4 s_3 s_1 s_4 s_3,$$

$$r_4^{\boldsymbol{w}} = s_3 s_4 s_1 s_3 s_4 (s_3 s_1)^2 s_3 s_4 s_3 (s_1 s_3)^2 s_4 s_3 s_1 s_4 s_3.$$

The $l$-vectors are

$$\begin{aligned}
l_1^{\boldsymbol{w}} &= s_3 s_4 s_3(\alpha_1) = (1, 0, -1, -1),\\
l_2^{\boldsymbol{w}} &= s_3 s_4 s_3 s_1 s_3 s_4(\alpha_2) = (-1, 1, 0, 1),\\
l_3^{\boldsymbol{w}} &= s_3 s_4 s_1 s_3 s_4 s_3 s_1(\alpha_3) = (2, 0, 0, -3),\\
l_4^{\boldsymbol{w}} &= s_3 s_4 s_1 s_3 s_4 (s_3 s_1)^2 s_3(\alpha_4) = (-3, 0, 0, 4),
\end{aligned}$$

and the $L$-matrix is given by

$$L^{\boldsymbol{w}} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ -1 & 1 & 0 & 1 \\ 2 & 0 & 0 & -3 \\ -3 & 0 & 0 & 4 \end{bmatrix}.$$

- From the construction, the $l$-vectors are roots of the equation

$$q(x_1, \dots, x_n) = 1.$$

  Since we do not yet have an interpretation of $l$-vectors as real roots of a "root system", we call them real Lösungen.
  (A Lösung means a solution in German.)

- Q1: How much information do the $l$-vectors carry?
  In particular, if $C^{\mathbf{v}} = C^{\mathbf{w}}$, is it true that $L^{\mathbf{v}} = L^{\mathbf{w}}$?

- Q2: Can we characterize the sequences $(i_1, \dots, i_r)$ for

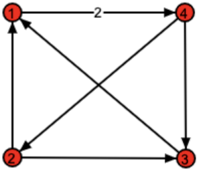
$$l_i^{\mathbf{w}} = s_{i_1} \cdots s_{i_r} \alpha_i ?$$

## Conjecture

*For any quiver Q, there exists an ordering $\prec$ such that, whenever $C^{\boldsymbol{v}} = C^{\boldsymbol{w}}$, we have*

$$r_i^{\boldsymbol{v}} = r_i^{\boldsymbol{w}} \text{ and } l^{\boldsymbol{v}} = \pm l^{\boldsymbol{w}}, \qquad i = 1, \ldots, n.$$

- We can show that $l$-vectors are leading terms in a certain sense when $c$-vectors are written as linear combinations of real Lösungen.
- This conjecture claims that these leading terms carry essential information about the $c$-vectors and quiver mutations.




$$\boldsymbol{w} = [3, 4, 1, 3, 4, 3], \qquad \boldsymbol{v} = [4, 1, 3, 4, 1, 3]$$

One can check

$$C^{\boldsymbol{w}} = C^{\boldsymbol{v}}.$$

Take $1 \prec 2 \prec 3 \prec 4$.

$$r_1^{\boldsymbol{w}} = s_3 s_4 s_3 s_1 s_3 s_4 s_3,$$

$$r_2^{\boldsymbol{w}} = s_3 s_4 s_3 s_1 s_3 s_4 s_2 s_4 s_3 s_1 s_3 s_4 s_3,$$

$$r_3^{\boldsymbol{w}} = s_3 s_4 s_1 s_3 s_4 s_3 s_1 s_3 s_1 s_3 s_4 s_3 s_1 s_4 s_3,$$

$$r_4^{\boldsymbol{w}} = s_3 s_4 s_1 s_3 s_4 (s_3 s_1)^2 s_3 s_4 s_3 (s_1 s_3)^2 s_4 s_3 s_1 s_4 s_3,$$

$$r_1^{\boldsymbol{v}} = s_3 (s_4 s_1)^2 s_4 s_3 s_4 s_1 s_4 s_3 s_4 (s_1 s_4)^2 s_3,$$

$$r_2^{\boldsymbol{v}} = s_3 (s_4 s_1)^2 s_4 s_3 s_4 s_1 s_4 s_3 (s_4 s_1)^2 s_4 s_2 s_4 (s_1 s_4)^2 s_3 s_4 s_1 s_4 s_3 s_4 (s_1 s_4)^2 s_3,$$

$$r_3^{\boldsymbol{v}} = s_3 (s_4 s_1)^2 s_4 s_3 s_4 (s_1 s_4)^2 s_3,$$

$$r_4^{\boldsymbol{v}} = (s_3 s_4 s_1)^2 s_4 (s_1 s_4 s_3)^2.$$

One can check

$$r_i^{\boldsymbol{w}} = r_i^{\boldsymbol{v}}, \quad i = 1, 2, 3, 4.$$

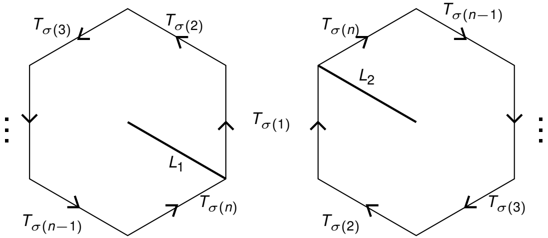
Recall Q2:

How can we describe the special sequences $(i_1, \ldots, i_r)$ for

$$l_i^{\boldsymbol{w}} = s_{i_1} \cdots s_{i_r} \alpha_i?$$

We will use curves on a Riemann surface.

- For $\sigma \in S_n$, consider



- Let $\Sigma_\sigma$ be the compact Riemann surface of genus $\lfloor \frac{n-1}{2} \rfloor$ obtained by gluing together the two $n$-gons with all the edges of the same label identified according to their orientations.
- $\mathcal{T} := T_1 \cup \cdots T_n \subset \Sigma_\sigma$

  $V$: the set of the vertex (or vertices) on $\mathcal{T}$
- $\mathfrak{W}$: the set of words $i_1 i_2 \cdots i_r$ from the alphabet $\{1, 2, \dots, n\}$ such that $i_p \neq i_{p+1}$,

  $\mathfrak{R} \subset \mathfrak{W}$: the set of words in $\mathfrak{W}$ such that $r$ is odd and $i_p = i_{r-p+1}$.

## Definition

An admissible curve $\eta : [0, 1] \longrightarrow \Sigma_\sigma$ is a curve such that

1) $\eta(x) \in V$ if and only if $x \in \{0, 1\}$;

2) there exists $\epsilon > 0$ such that $\eta([0, \epsilon]) \subset L_1$ and $\eta([1 - \epsilon, 1]) \subset L_2$;

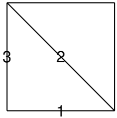
3) if $\eta(x) \in \mathcal{T} \setminus V$ then $\eta([x - \epsilon, x + \epsilon])$ meets $\mathcal{T}$ transversally for sufficiently small $\epsilon > 0$;

4) $\upsilon(\eta) \in \mathfrak{R}$, where $\upsilon(\eta) := i_1 \cdots i_k \in \mathfrak{W}$ is given by

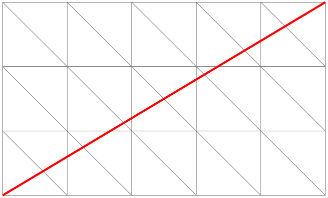
$$\{x \in [0, 1] \ : \ \eta(x) \in \mathcal{T}\} = \{x_1 < \cdots < x_k\} \quad \text{and}$$

$$\eta(x_\ell) \in T_{i_\ell} \text{ for } \ell \in \{1, ..., k\}.$$

For example, let $n = 3$ and $\sigma = (2, 3, 1)$. Then $\Sigma_\sigma$ is a triangulated torus with edges indexed by $1, 2, 3$ as follows:




Consider the universal cover of $\Sigma_\sigma$ and a curve $\eta$



We get $\upsilon(\eta) = 2321232321232$ for $\eta$.

Since $\upsilon(\eta) \in \mathfrak{R}$, the curve $\eta$ is admissible.

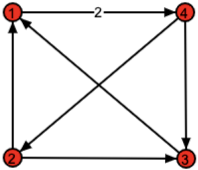
For an admissible curve $\eta$ with $\upsilon(\eta) = i_1 i_2 \cdots i_r$, define

$$s(\eta) = s_{i_1} s_{i_2} \cdots s_{i_r}.$$

## Conjecture

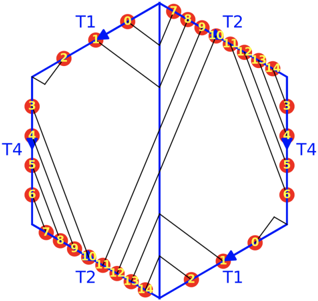
*Fix an ordering $\prec$ for a quiver Q. Then there exist non-self-crossing admissible curves $\eta_i^{\mathbf{w}}$, $i = 1, 2, \ldots, n$, on the Riemann surface $\Sigma_\sigma$ for some $\sigma \in S_n$ such that*

$$r_i^{\mathbf{w}} = s(\eta_i^{\mathbf{w}}), \quad i = 1, 2, \ldots, n.$$

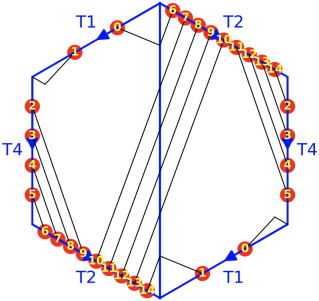



$$\mathbf{v} = [2, 3, 4, 2, 1, 3]$$

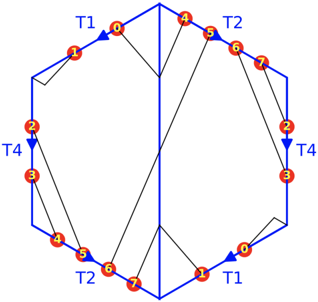
Take $1 \prec 3 \prec 2 \prec 4$ and $\sigma = (3, 1, 4, 2)$.



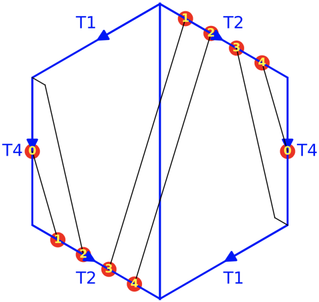

(A) The curve $\eta_1^{\boldsymbol{v}}$.




(B) The curve $\eta_2^{\boldsymbol{v}}$.




(C) The curve $\eta_3^{\boldsymbol{v}}$.




(D) The curve $\eta_4^{\boldsymbol{v}}$.

# Thank You