Singularities of Schubert varieties and Nakajima's quiver varieties

(Some Problems about Unimodality)

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What is a unimodal sequence?

A sequence of integers $a_0, a_1, \ldots, a_n$ (or the polynomial $a_0 + a_1 x + \cdots + a_n x^n$) is unimodal if there exists $0 \leq i \leq n$ such that $a_0 \leq a_1 \leq \cdots \leq a_i \geq a_{i+1} \geq \cdots \geq a_n$.

Example

$(1, 5, 10, 10, 5, 1)$, $(1, 1, 1, 3, 2)$, $(1, 1, 2, 3, 4)$, $(4, 3, 2, 1, 1)$, $(1, 1, 1, 1, 1)$.

Richard Stanley (1989): Log-concave and unimodal sequences in algebra, combinatorics, and geometry.

Recently June Huh and his collaborators made breakthrough by relating this with Hodge theory.

Part I. From Schubert Varieties

Young tableaux

Definition

Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$, identity it with its Young diagram (a left-justified shape of k-rows of boxes of lengths $\lambda_1, \ldots, \lambda_k$).

An Increasing Young tableau (IYT) assigns to each box a positive integer, such that the entries strictly increase along rows and columns.

Example

$$T = \begin{array}{|c|c|c|c|} \hline 2 & 4 & 6 & 7 \\ \hline 3 & 7 & 8 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} \end{array}$$ is a IYT of shape $\lambda = (4, 3, 1)$.

Young tableaux

Definition

For an IYT T, assume $T(0,+) = 0$, $T(+,0) = 0$, and define

$$\mathrm{depth}(T) = \sum_{i,j} \Big(T(i,j) - 1 - \max\big(T(i,j-1), T(i-1,j)\big) \Big)$$

Example

For $T = \begin{array}{|c|c|c|c|} \hline 2 & 4 & 6 & 7 \\ \hline 3 & 7 & 8 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} \end{array}$, add 0s to the left column and the top row: $\begin{array}{c|c|c|c|c|} \cline{2-5} & 0 & 0 & 0 & 0 \\ \hline \multicolumn{1}{|c|}{0} & 2 & 4 & 6 & 7 \\ \hline \multicolumn{1}{|c|}{0} & 3 & 7 & 8 & \\ \cline{1-4} \multicolumn{1}{|c|}{0} & 4 & & & \\ \cline{1-2} \end{array}$

$\mathrm{depth}(T)$ = sum of

$$\begin{bmatrix} 1 - \max(0,0) = 1 & 3 - \max(2,0) = 1 & 5 - \max(4,0) = 1 & 6 - \max(6,0) = 0 \\ 2 - \max(0,2) = 0 & 6 - \max(3,4) = 2 & 7 - \max(7,6) = 0 & \\ 3 - \max(0,3) = 0 & & & \end{bmatrix} = 5$$

In other words, we can insert $d(T) = 5$ smaller numbers (in red) to T to make it a 'saturated set-valued' increasing tableau: $\mathrm{sat}(T) = \begin{array}{|c|c|c|c|} \hline 12 & 34 & 56 & 7 \\ \hline 3 & 567 & 8 & \\ \cline{1-3} 4 & & & \\ \cline{1-1} \end{array}$

Unimodality

Question (1)

Let λ be a partition of length ℓ, let $\mathbf{b} = (b_1, \ldots, b_\ell)$ be a sequence of positive integers. Define a polynomial in $\mathbb{Z}[q]$ as

$$H_{\lambda,\mathbf{b}}(q) = \sum_T q^{\mathrm{depth}(T)}$$

where T runs over all IYT of shape λ and all entries in the i-th row is $\leq b_i$, for $i = 1, \ldots, \ell$. Is $H_{\lambda,\mathbf{b}}(q)$ always unimodal?

Example

Let $\lambda = (2,2)$, $\mathbf{b} = (3,4)$. There are 6 such IYT:

$$T = \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$\mathrm{depth}(T) = \quad 0 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1$$

So $H_{\lambda,\mathbf{b}}(q) = 1 + 4q + q^2$.

Note that $H_{\lambda,\mathbf{b}}(q)$ is not always symmetric. For $\lambda = (3,2)$, $\mathbf{b} = (4,5)$. $H_{\lambda,\mathbf{b}}(q) = 1 + 6q + 3q^2$.

Algebra origin : Hilbert-Samuel Multiplicity

Definition

Let p be a point on a variety V (over $\mathbb{C}$) of dim d, with local ring $(\mathcal{O}_p, \mathfrak{m})$.
For $n \gg 0$, $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1}$ coincides with a unique polynomial

$$P(n) = a_{d-1}n^{d-1} + \cdots + a_1 n + a_0 \in \mathbb{Q}[n].$$

Define the Hilbert-Samuel multiplicity of p to be

$$\mathrm{mult}_p V = (d-1)!\, a_{d-1}$$

Example

Hilbert-Samuel multiplicity of $p = (0,0,0)$ in the cone $z^3 = x^3 + y^3$ in $\mathbb{C}^3$.
The sequence $\dim \mathfrak{m}^n/\mathfrak{m}^{n+1}$ for $n = 0, 1, 2, \ldots$ is:

$$1, 3, 6, 9, 12, 15, \ldots$$

The polynomial $P(n) = 3n$, therefore $\mathrm{mult}_p(V) = 3$.

Hilbert-Samuel multiplicity is refined by the "H-polynomial" $H_p \in \mathbb{Z}[q]$, the numerator of the Hilbert series of $\mathrm{gr}_{\mathfrak{m}_p}\mathcal{O}_p$:

$$\frac{H_p}{(1-q)^d} = \mathrm{Hilb}(\mathrm{gr}_{\mathfrak{m}_p}\mathcal{O}_p) = \sum_{n=0}^{\infty} \dim(\mathfrak{m}_p^n/\mathfrak{m}_p^{n+1})q^n$$

We view H_p as an analogue of Hilbert-Samuel multiplicity since $H_p(1) = \mathrm{mult}_p V$.

Example (Continue previous example)

$$\frac{H_p}{(1-q)^2} = 1 + 3q + 6q^2 + 9q^3 + 12q^4 + \cdots = \frac{1+q+q^2}{(1-q)^2}$$

We get $H_p = 1 + q + q^2$, a "refinement" of the multiplicity 3.

Consider the Schubert variety $X_w = \overline{Be_w} \subseteq \mathrm{Flags}(\mathbb{C}^n) = \mathrm{GL}_n/B$, and the point $e_v \in X_w$ (where $v \leq w \in S_n$ in Bruhat order).

Assume $w \in S_n$ is covexillary (i.e. 3412-avoiding: no indices $i_1 < i_2 < i_3 < i_4$ such that $w(i_1), w(i_2), w(i_3), w(i_4)$ are in the same order as 3412.)

Theorem (L-, Alex Yong)

Let $w \in S_n$ be covexillary. Then the H-polynomial of the point e_v in X_w is equal to $H_{\lambda,\mathbf{b}}(q)$ for some λ and $\mathbf{b}$ explicitly determined by v, w.

A negative answer to Question (1) (if there is an example of $H_{\lambda,\mathbf{b}}(q)$ that is not unimodal) will probably disprove a conjecture by Stanley in 1989:

Any graded Cohen-Macualy domain R over a field generated by R_1 is unimodal.

Some Variations of Question (1)

Question (2)

Let λ be a partition of length ℓ, let $\mathbf{a} = (a_{ij})$, $\mathbf{b} = (b_{ij})$ be two arrays of positive integers. Define a polynomial in $\mathbb{Z}[q]$ as

$$H_{\lambda,\mathbf{a},\mathbf{b}}(q) = \sum_{T} q^{\mathrm{depth}(T)}$$

where T runs over all IYT of shape λ and $a_{ij} \leq T(i,j) \leq b_{ij}$.
Is $H_{\lambda,\mathbf{a},\mathbf{b}}(q)$ always unimodal?

Example

$\lambda = (2,1)$, bounds given by $\begin{bmatrix} 1 \leq T(1,1) \leq 3 & 3 \leq T(1,2) \leq 4 \\ 3 \leq T(2,1) \leq 4 & \end{bmatrix}$

Possible IYT are:

$$T = \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 4 \\ \cline{1-1}\end{array}$$

$\mathrm{depth}(T) =$	2	3	3	4	1	2	2	3	2

So $H_{\lambda,\mathbf{a},\mathbf{b}}(q) = q + 4q^2 + 3q^3 + q^4$

Some Variations of Question (1)

What happens if we replace $\mathrm{depth}(T)$ by the $\mathrm{sum}(T)$, the sum of all entries in T?

Question (3)

Let λ be a partition of length ℓ, let $\mathbf{a} = (a_{ij})$, $\mathbf{b} = (b_{ij})$ be two arrays of positive integers. Define a polynomial in $\mathbb{Z}[q]$ as

$$H'_{\lambda,\mathbf{a},\mathbf{b}}(q) = \sum_T q^{\mathrm{sum}(T)}$$

where T runs over all IYT of shape λ and $a_{ij} \le T(i,j) \le b_{ij}$.
Is $H'_{\lambda,\mathbf{a},\mathbf{b}}(q)$ always unimodal?

Example

$\lambda = (2,1)$, bounds given by $\begin{bmatrix} 1 \le T(1,1) \le 3 & 3 \le T(1,2) \le 4 \\ 3 \le T(2,1) \le 4 & \end{bmatrix}$

Possible IYT are:

$$T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 4 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 4 \\ \hline 4 \\ \cline{1-1}\end{array}$$

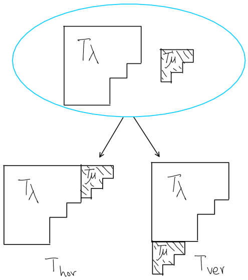
$$\mathrm{sum}(T) = \quad 7 \quad 8 \quad 8 \quad 9 \quad 8 \quad 9 \quad 9 \quad 10 \quad 11$$

So $H'_{\lambda,\mathbf{a},\mathbf{b}}(q) = q^7 + 3q^8 + 3q^9 + q^{10} + q^{11}$

Part II. From Quiver Varieties

Determinantal varieties and beyond

Given an IYT T_λ of shape λ, and Y_μ of shape μ, we glue them together horizontally as T_{hor} or vertically as T_{ver} (if possible).



Impose boundary conditions and we can construct $H_{\lambda,\mu,\mathbf{b}}(q)$ as before.

Question (4)

Is $H_{\lambda,\mu,\mathbf{b}}(q)$ always unimodal?

Example

Let $\lambda = (2,2)$, $\mu = (1,1)$. Impose a upper bound 5 to all entries. Then

$$T_\lambda = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline \end{array} \qquad T_\mu = \begin{array}{|c|}\hline e \\ \hline f \\ \hline \end{array} \qquad \Longrightarrow \qquad T_{\text{hor}} = \begin{array}{|c|c|c|}\hline a & b & e \\ \hline c & d & f \\ \hline \end{array} \qquad T_{\text{ver}} = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline e \\ \cline{1-1} f \\ \cline{1-1} \end{array}$$

In order for T_{hor} and T_{ver} to be valid IYTs, we must have

$$a < b < e, \quad c < d < f, \quad a < c < e < f, \quad b < d$$

The possible T_{hor} and corresponding $\operatorname{depth}(T)$ are:

$$\begin{array}{cccccccc} T = & \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 4 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 4 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array} & \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 3 & 4 & 5 \\ \hline \end{array} \\ \operatorname{depth}(T) = & 0 & 1 & 1 & 1 & 1 & 2 & 1 \end{array}$$

So $H_{\lambda,\mu,\mathbf{b}}(q) = 1 + 5q + q^2$

Determinantal varieties are well-known examples in algebraic geometry and are intensively studied.

Example

$I =$ ideal generated by all 2-minors of a 2×3 matrix of indeterminates $\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$
$= (ae - bd, af - cd, bf - ce)$.

The H-polynomial of $V(I)$ (the variety defined by I) at the origin p is

$$H_p = 1 + 2q$$

In general, for 2-minors:

matrix size	2	3	4	5
2	$1 + q$	$1 + 2q$	$1 + 3q$	$1 + 4q$
3		$1 + 4q + q^2$	$1 + 6q + 3q^2$	$1 + 8q + 6q^2$
4			$1 + 9q + 9q^2 + q^3$	$1 + 12q + 18q^2 + 4q^3$
5				$1 + 16q + 36q^2 + 16q^3 + q^4$

It is known that the minors for a Groebner basis for I under a diagonal order.

Then standard algebraic manipulation shows H-polynomial can be computed using "pipe dreams" (a minimal set poisoning diagonals of all minors), or equivalently, using IYT we studied before:

$$\text{Pipe dream} = \quad \begin{array}{ccc} + & + & \cdot \\ + & + & \cdot \\ \cdot & \cdot & \cdot \end{array} \quad \begin{array}{ccc} + & + & \cdot \\ + & \cdot & \cdot \\ \cdot & \cdot & + \end{array} \quad \begin{array}{ccc} + & + & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & + & + \end{array} \quad \begin{array}{ccc} + & \cdot & \cdot \\ + & \cdot & + \\ \cdot & \cdot & + \end{array} \quad \begin{array}{ccc} + & \cdot & \cdot \\ \cdot & \cdot & + \\ \cdot & + & + \end{array} \quad \begin{array}{ccc} \cdot & \cdot & \cdot \\ \cdot & + & + \\ \cdot & + & + \end{array}$$

$$T = \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & 4 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$\mathrm{depth}(T) = \quad 0 \quad 1 \quad 1 \quad 1 \quad 2 \quad 1$$

Note that all pipe dreams can be obtained from the first one via the following move:

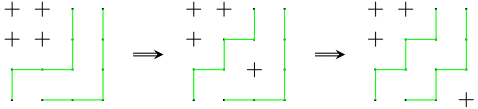
$$\begin{array}{cc} {\color{red}+} & \cdot \\ \cdot & \cdot \end{array} \quad \Longrightarrow \quad \begin{array}{cc} \cdot & \cdot \\ \cdot & + \end{array}$$

Determinantal varieties and beyond

The following combinatorial objects are bijective:

$$\{ \text{ IYT } \} \xleftrightarrow{1:1} \{ \text{ pipe dreams } \} \xleftrightarrow{1:1} \{ \text{ families of non-intersecting “paths”} \}$$

Example



Determinantal varieties and beyond

A generalization of determinantal varieties is the class of **double determinantal varieties**, which are special cases of Nakajima's quiver varieties, as well as the **subspace varieties** considered in [Landsberg, Tensors: Geometry and applications].

The following example gives the idea of the definition of double determinantal ideal:

Consider two "pages" of matrices of indeterminates

$$X_1 = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}, \quad X_2 = \begin{bmatrix} a' & b' \\ c' & d' \\ e' & f' \end{bmatrix}$$

combine them in two ways:

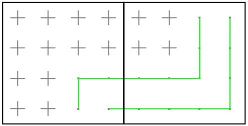
$$A = [X_1 \ X_2] = \begin{bmatrix} a & b & a' & b' \\ c & d & c' & d' \\ e & f & e' & f' \end{bmatrix} \quad B = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \\ e & f \\ a' & b' \\ c' & d' \\ e' & f' \end{bmatrix}$$

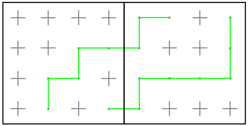
A double determinantal ideal is the ideal I generated by all 2-minors of A and B.

Theorem (Nathan Fieldsteel and Patricia Klein, Josua Illian and L–, Aldo Conca, Emanuela De Negri, Zeljka Stojanac (a less general statement))

The minors form a Groebner basis for the double determinantal ideal I under diagonal order.

With the help of the theorem, we can derives a combinatorial formula for the H-polynomial. So the story is similar as before. For example,





This gives the algebra motivation of Question (4).

Motivation: Nakajima's graded quiver varieties and cluster algebras

Quiver: $1 \xleftarrow{r} 2$

Vector spaces with dimension $w = (w_1', w_2)$, $v = (v_1, v_2)$:

$$W = W_1' \oplus W_2', \quad V = V_1 \oplus V_2$$

Space of quiver representations:

$$\begin{tikzcd} 0 & V_2 \arrow[l, "\beta_2"'] \arrow[d, shift left, "b_1,\dots,b_r"] \arrow[d, shift right] & W_2 \arrow[l, "\alpha_2"'] \\ W_1' & V_1 \arrow[l, "\beta_1"'] & 0 \arrow[l, "\alpha_1"'] \end{tikzcd}$$

Define two versions of Nakajima's quiver varieties:

(a) Affine graded quiver variety $\mathcal{M}_0^\bullet(V, W) = \{(\alpha_i, \beta_i, b_j)\} // \prod GL(V_i)$

(b) Nonsingular graded quiver variety $\mathcal{M}^\bullet(V, W) = \{(\alpha_i, \beta_i, b_j)\}^s / \prod GL(V_i)$

Motivation: Nakajima's graded quiver varieties and cluster algebras

Alternatively, forget about V_1, V_2,

$$\begin{tikzcd} 0 & W_2 \arrow[l] \arrow[dl, shift left, "y_h=\beta_1 b_h \alpha_2 \ \ (h=1,\dots,r)"] \arrow[dl, shift right] \\ W_1' & 0 \arrow[l] \end{tikzcd} \quad .$$

Define

$$y_1 + \cdots + y_r : W_2^{\oplus r} \to W_1'$$
$$y_1 \oplus \cdots \oplus y_r : W_2 \to (W_1')^{\oplus r}$$

In terms of matrices:

$$A(\mathbf{y}) = \begin{bmatrix} y_1 & \cdots & y_r \end{bmatrix}, \quad B(\mathbf{y}) = \begin{bmatrix} y_1 \\ \vdots \\ y_r \end{bmatrix}$$

The space of quiver representations is $\mathbf{E}_w := \bigoplus_{h=1}^r \mathrm{Hom}(W_2, W_1')$.

$\mathcal{M}_0^\bullet(v, w)$ is isomorphic to

$$\mathbf{E}_{v,w} := \{(y_1, \ldots, y_r) \in \mathbf{E}_w \mid \mathrm{rank} A(\mathbf{y}) \leq v_1, \mathrm{rank} B(\mathbf{y}) \leq v_2\}.$$

which is the double determinantal variety!

Part III. The Major Index

Consider the set of standard Young tableaux (SYT) of shape λ (which is an IYT filled with $1, 2, \ldots, |\lambda|$, each number appears exactly once).
The Major index of a SYT T is

$$\mathrm{maj}(T) = \sum i$$

for those i such that $i + 1$ lies in a row strictly below the cell containing i.
Define the major index generating function for λ to be

$$\mathrm{SYT}^{\mathrm{maj}}(q) = \sum_{T} q^{\mathrm{maj}(T)}$$

S. Billey asked the following question during a talk in Fomin's 60th birthday conference (2018).[1]

Question (5)

For which λ are $\mathrm{SYT}^{\mathrm{maj}}(q)$ unimodal?

[1] https://sites.math.washington.edu/ billey/talks/michigan.pdf

Example

For $\lambda = (3,2,1)$, $T = \begin{array}{|c|c|c|} \hline \boxed{1} & 3 & \boxed{4} \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} \end{array}$, we have $\mathrm{maj}(T) = 1 + 4 = 5$.

Run through all T, we get

$$\mathrm{SYT}^{\mathrm{maj}}(q) = q^{11} + 2q^{10} + 2q^9 + 3q^8 + 3q^7 + 2q^6 + 2q^5 + q^4$$

A question posed by Sara Billey

Billey's conjecture. (A corner is a cell of hook length 1)

Conjecture. The polynomial $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal if λ has at least 4 corners. If λ has 3 corners or fewer, then $\mathrm{SYT}^{\mathrm{maj}}(q)$ is unimodal except when λ or λ' is among the following partitions:

1. Any partition of rectangle shape that has more than one row and column.
2. Any partition of the form $(k, 2)$ with $k \geq 4$ and k even.
3. Any partition of the form $(k, 4)$ with $k \geq 6$ and k even.
4. Any partition of the form $(k, 2, 1, 1)$ with $k \geq 2$ and k even.
5. Any partition of the form $(k, 2, 2)$ with $k \geq 6$.
6. Any partition on the list of 40 special exceptions of size at most 28.

Special Exceptions.

$$(3,3,2),(4,2,2),(4,4,2),(4,4,1,1),$$
$$(5,3,3),(7,5),(6,2,1,1,1,1),$$
$$(5,5,2),(5,5,1,1),(5,3,2,2),(4,4,3,1),$$
$$(4,4,2,2),(7,3,3),(8,6),(6,6,2),$$
$$(6,6,1,1),(5,5,2,2),(5,3,3,3),(4,4,4,2),$$
$$(11,5),(10,6),(9,7),(7,7,2),$$
$$(7,7,1,1),(6,6,4),(6,6,1,1,1,1),(6,5,5),$$
$$(5,5,3,3),(12,6),(11,7),(10,8),$$
$$(15,5),(14,6),(11,9),(16,6),(12,10),(18,6),$$
$$(14,10),(20,6),(22,6).$$

But there are too many exceptional cases in the conjecture ...

Question (6)

Is there a better/refined claim for the unimodality of $\mathrm{SYT}^{\mathrm{maj}}(q)$*?*

Let us consider the first exceptional case.

Example

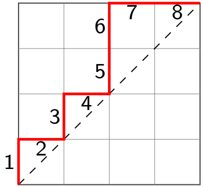
A partition of two-row rectangle shape (n, n). There is a bijection

$$\{SYT \text{ of shape } (n,n)\} \overset{1:1}{\longleftrightarrow} \{\text{ Dyck paths in } n \times n\}$$

the i-th edge of the Dyck path is in the $\begin{cases} \text{1st row of the SYT if it is vertital} \\ \text{2nd row of the SYT if it is horizontal} \end{cases}$

1	3	5	6
2	4	7	8

$\longleftrightarrow$



Under this bijection, $\text{maj}(\text{SYT}) = \text{maj}(\text{Dyck path}) + n$

The MacMahon q-Catalan number is well studied:

$$C_n(q) = \sum q^{\text{maj(Dyck path)}}$$

A question posed by Sara Billey

An example of $SYT^{\mathrm{maj}}(q)$ for $n = 4$:

$$\mathrm{SYT}^{\mathrm{maj}}(q) = q^4(q^{12}+q^{10}+q^9+2q^8+q^7+2q^6+q^5+2q^4+q^3+q^2+1) = q^4 \cdot \frac{1}{[n+1]} \frac{[2n]!}{[n]!}.$$

The coefficients $(1, 0, 1, 1, 2, 1, 2, 1, 2, 1, 1, 0, 1)$ are symmetric but not unimodal.

Nevertheless, the coefficients of even and odd powers are unimodal.

$$\text{Even: } (1, 1, 2, 2, 2, 1, 1) \qquad \text{Odd: } (0, 1, 1, 1, 1, 1, 0)$$

This property is called "parity-unimodality".

Eric Stucky (2018)[2] shows that MacMahon q-Catalan numbers (and much more) are parity-unimodal.

In the recent update (May 2020) of the paper by Billey, Konvalinka, Swanson[3] they add a section conjecturing that $\mathrm{SYT}^{\mathrm{maj}}(q)$ is parity-unimodal.

[2] Cyclic Sieving, Necklaces, and Bracelets, arXiv:1812.04578

[3] Tableau posets and the fake degrees of coinvariant algebras, arXiv:1809.07386

A question posed by Sara Billey

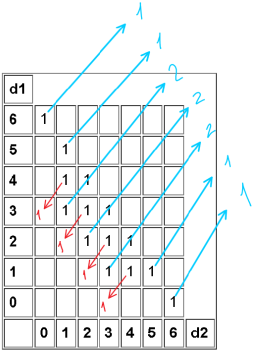
This parity-unimodality of MacMahon q-Catalan numbers can be implied by:

Question (7)

Let $C_n(q,t) = \sum e_{d_1,d_2} q^{d_1} t^{d_2}$ be a q,t-Catalan number. Are the coefficients $(e_{d,0}, e_{d-1,1}, \ldots, e_{0,d})$ unimodal?

For example, look at the table of coefficients of $C_4(q,t)$.

d1								
6	1							
5		1						
4		1	1					
3		1	1	1				
2			1	1	1			
1				1	1	1		
0							1	
	0	1	2	3	4	5	6	d2



(It is also conjectured that each row and column of the coefficients table is unimodal. And similar conjecture for rational q,t-Catalan numbers.)

A question posed by Sara Billey

So Billey's question can be refined as:

Is there a two-variable version of $\mathrm{SYT}^{\mathrm{maj}}(q)$ *in the form*

$$\mathrm{SYT}^{?,?}(q,t) = \sum_T q^{?(T)} t^{?(T)}$$

so that the parity-unimodality is a consequence of the unimodality of each graded piece $\mathrm{SYT}^{?,?}(q,t)$*?*

A question to start with:

For $\lambda = (n,n,n)$, $\mathrm{SYT}^{\mathrm{maj}}(q)$ *is a 3-dimensional analogue of a MacMahon* q*-Catalan number. Can we find a 3-dimensional analogue of a* q,t*-Catalan number?*

The End.

Thank you!