

Robinson-Schensted correspondence for natural unit interval orders

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(joint with Pavlo Pylyavskyy)

Outline

- 1 What does the title mean?
- 2 Motivation
- 3 \mathcal{P} -Knuth equivalence
- 4 Ladders and the main theorem
- 5 \mathcal{P} -Robinson-Schensted algorithm

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Robinson-Schensted correspondence

Robinson-Schensted-Knuth correspondence is a bijection

$$\{\mathbb{N}\text{-valued matrices}\} \rightarrow \bigsqcup_{\lambda} \text{SSYT}_{\lambda} \times \text{SSYT}_{\lambda}$$

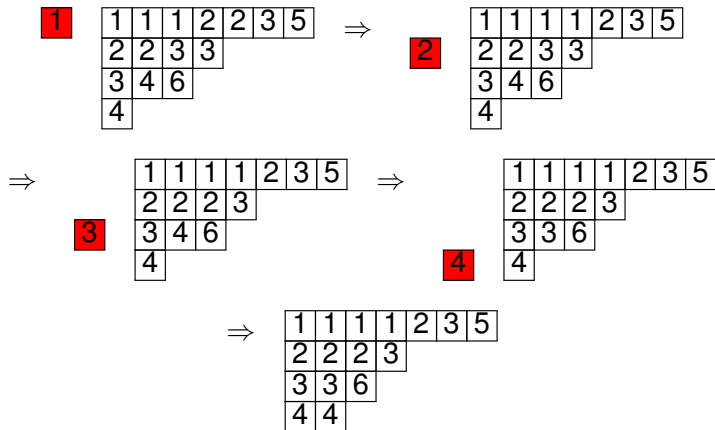
that is usually described in terms of “bumping process”.

If we restrict the domain to permutation matrices, then it becomes a bijection

$$\mathcal{S}_n \rightarrow \bigsqcup_{\lambda \vdash n} \text{SYT}_{\lambda} \times \text{SYT}_{\lambda}$$

which is the usual Robinson-Schensted correspondence.

Robinson-Schensted correspondence



Natural unit interval orders

Suppose that a partial order \mathcal{P} on $[1, n]$ is given. We assume that

$$a \succ_{\mathcal{P}} b \Rightarrow a > b,$$

i.e. the usual order on $[1, n]$ is a linearization of \mathcal{P} . We write:

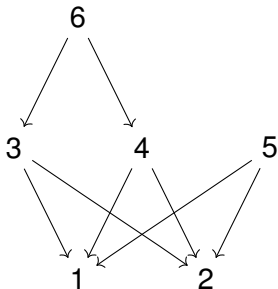
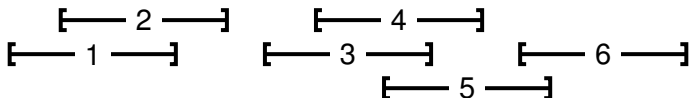
- $a \rightarrow_{\mathcal{P}} b$ if a is bigger than b with respect to \mathcal{P} ,
- $a \dashrightarrow_{\mathcal{P}} b$ if a and b are not comparable in \mathcal{P} , and
- $a \dashrightarrow_{\mathcal{P}} b$ if $a > b$ and $a \dashrightarrow_{\mathcal{P}} b$.

Definition

We say that \mathcal{P} is a natural unit interval order if for any $a, b, c \in [1, n]$ such that $a \rightarrow_{\mathcal{P}} c$, $a \dashrightarrow_{\mathcal{P}} b$, and $b \dashrightarrow_{\mathcal{P}} c$, we have $a > b > c$.

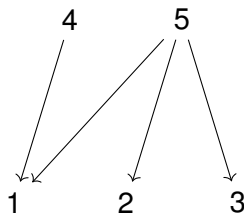
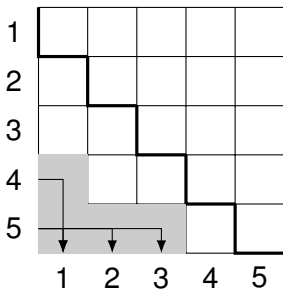
Natural unit interval orders

If \mathcal{P} is a natural unit interval order on $[1, n]$, then there exist $y_1, y_2, \dots, y_n \in \mathbb{R}$ such that $y_1 < y_2 < \dots < y_n$ and $a \rightarrow_{\mathcal{P}} b \Leftrightarrow y_a > y_b + 1$.



Natural unit interval orders

If \mathcal{P} is a natural unit interval order on $[1, n]$, then there exists a partition $\lambda \subset (n-1, n-2, \dots, 2, 1)$ such that $a \leftarrow_{\mathcal{P}} b$ if and only if $a \leq \lambda_{n+1-b}$. In this case we write $\mathcal{P} = \mathcal{P}_{\lambda, n}$.



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Stanley-Stembridge conjecture

Recall the fundamental quasisymmetric function F_S for $S \in [1, n-1]$ defined by

$$F_S = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_a < i_{a+1} \Leftrightarrow a \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

For $w \in \mathcal{S}_n$, we define its \mathcal{P} -descent to be

$$\text{des}_{\mathcal{P}}(w) = \{i \in [1, n-1] \mid w_i \rightarrow_{\mathcal{P}} w_{i+1}\}.$$

We consider the following quasisymmetric function

$$\gamma_{\mathcal{P}, \mathcal{S}_n} = \sum_{w \in \mathcal{S}_n} F_{\text{des}_{\mathcal{P}}(w)}.$$

For example, if \mathcal{P} is the usual order then $\gamma_{\mathcal{P}, \mathcal{S}_n} = h_{1^n}$.

Stanley-Stembridge conjecture

Lemma

$\gamma_{\mathcal{P}, S_n}$ is a symmetric function and p -positive.

Theorem (Haiman, Gasharov)

$\gamma_{\mathcal{P}, S_n}$ is Schur-positive.

Conjecture (Stanley, Stembridge)

$\gamma_{\mathcal{P}, S_n}$ is h -positive.

This conjecture is still open, but partial progress was made: Stanley-Stembridge, Gebhard-Sagan, Dahlberg-van Willigenburg, Harada-Precup, Cho-Huh, Cho-Hong, etc.

Graded Stanley-Stembridge conjecture

For $w \in \mathcal{S}_n$, set its “fake \mathcal{P} -inversion” to be

$$\text{f-inv}_{\mathcal{P}}(w) = \{(w_i, w_j) \in [1, n]^2 \mid i < j, w_i \dashrightarrow_{\mathcal{P}} w_j\}.$$

We consider the following weighted version

$$\tilde{\gamma}_{\mathcal{P}, \mathcal{S}_n} = \sum_{w \in \mathcal{S}_n} t^{|\text{f-inv}_{\mathcal{P}}(w)|} F_{\text{des}_{\mathcal{P}}(w)}.$$

Theorem (Shareshian-Wachs)

$\tilde{\gamma}_{\mathcal{P}, \mathcal{S}_n}$ is a symmetric function and both p - and Schur-positive.

Conjecture (Shareshian-Wachs)

$\tilde{\gamma}_{\mathcal{P}, \mathcal{S}_n}$ is h -positive.

Hessenberg varieties

For $\lambda \subset (n-1, \dots, 2, 1)$ and regular semisimple $s \in GL_n$, we let

$$\mathcal{H}_{\text{ess}_{\lambda,s}} = \{F_{\bullet} = [F_0 \subset F_1 \subset \dots \subset F_{n-1} \subset F_n] \mid s \cdot F_i \subset F_{n-\lambda'_i}\}$$

called a Hessenberg variety.

Tymoczko defined a so-called “dot-action” on $\bigoplus_{i \in \mathbb{Z}} H_T^{2i}(\mathcal{H}_{\text{ess}_{\lambda,s}}) t^i$ where T is the maximal torus containing s , which makes it into a graded \mathcal{S}_n -module.

Theorem (Brosnan-Chow, Guay-Paquet)

The Frobenius character of $\bigoplus_{i \in \mathbb{Z}} H^{2i}(\mathcal{H}_{\text{ess}_{\lambda,s}}) t^i$ equals $\tilde{\gamma}_{\mathcal{P}_{\lambda,n}, \mathcal{S}_n}$.

This is originally conjectured by Shareshian-Wachs.

Goal

Our goal is to understand combinatorics behind this picture.

- Introduce \mathcal{P} -Knuth equivalence
- Define \mathcal{P} -Robinson-Schensted algorithm
- Use these combinatorial tools to analyze $\tilde{\gamma}_{\mathcal{P}, \lambda, n, \mathcal{S}_n}$ in detail

\rightsquigarrow refinement of the results of Shareshian-Wachs

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Knuth moves and equivalence

Regard \mathcal{S}_n as a set of words with alphabets in $[1, n]$.

Definition

We say that $w, w' \in \mathcal{S}_n$ are connected by a Knuth move if for $a, b, c \in [1, n]$ such that $a < b < c$ we have either

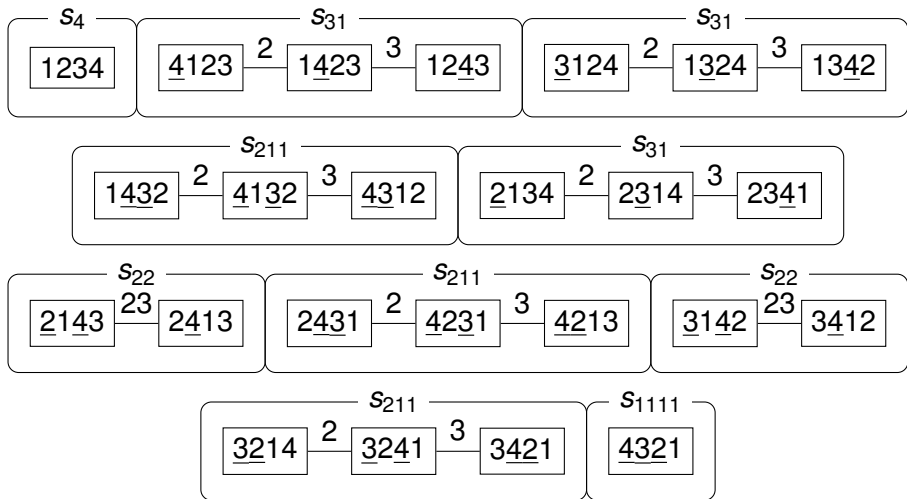
$$w = \dots cab \dots \iff w' = \dots acb \dots \text{ or}$$

$$w = \dots bca \dots \iff w' = \dots bac \dots .$$

The Knuth equivalence is defined to be the closure of these moves.

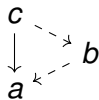
For each equivalence class Γ , consider the following generating function $\gamma_\Gamma = \sum_{w \in \Gamma} F_{\text{des}(w)}$. Then,

- $\gamma_{\mathcal{P}, \mathcal{S}_n} = \sum_\Gamma \gamma_\Gamma$ when \mathcal{P} is the usual order, and
- $\gamma_\Gamma = s_\lambda$ for some $\lambda \vdash n$.

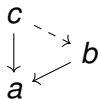
Example: \mathcal{S}_4 

\mathcal{P} -Knuth moves and equivalence

For $a, b, c \in [1, n]$ such that $a < b < c$ and $a \leftarrow_{\mathcal{P}} c$, there are four possibilities of $\mathcal{P}|_{\{a,b,c\}}$:



$$(1) \mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(1),3}$$



$$(2) \mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(1,1),3}$$



$$(3) \mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(2),3}$$



$$(4) \mathcal{P}|_{\{a,b,c\}} \simeq \mathcal{P}_{(2,1),3}$$

\mathcal{P} -Knuth moves and equivalence

In each case, we define the \mathcal{P} -Knuth move as follows:

- ① $a \leftarrow\text{---}\mathcal{P} b$ and $b \leftarrow\text{---}\mathcal{P} c$:

$$\dots \underline{bca} \dots \overset{\mathcal{P}}{\leftarrow\rightsquigarrow} \dots \underline{cab} \dots .$$

- ② $a \leftarrow\mathcal{P} b$ and $b \leftarrow\text{---}\mathcal{P} c$:

$$\dots \underline{bca} \dots \overset{\mathcal{P}}{\leftarrow\rightsquigarrow} \dots \underline{bac} \dots \quad \text{and} \quad \dots \underline{cba} \dots \overset{\mathcal{P}}{\leftarrow\rightsquigarrow} \dots \underline{cab} \dots .$$

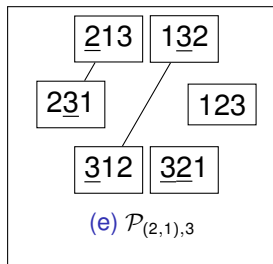
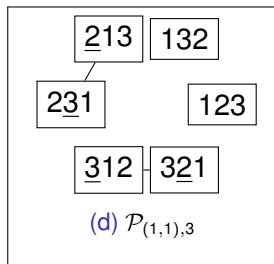
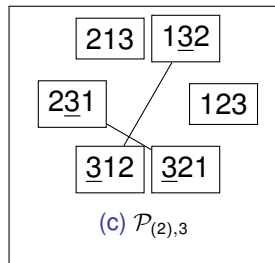
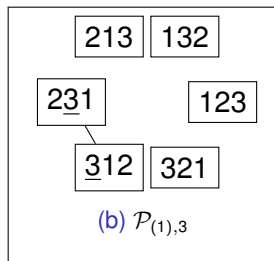
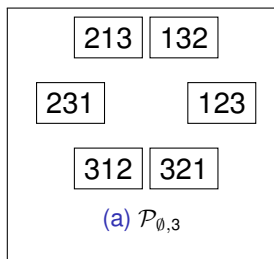
- ③ $a \leftarrow\text{---}\mathcal{P} b$ and $b \leftarrow\mathcal{P} c$:

$$\dots \underline{bca} \dots \overset{\mathcal{P}}{\leftarrow\rightsquigarrow} \dots \underline{cba} \dots \quad \text{and} \quad \dots \underline{acb} \dots \overset{\mathcal{P}}{\leftarrow\rightsquigarrow} \dots \underline{cab} \dots .$$

- ④ $a \leftarrow\mathcal{P} b \leftarrow\mathcal{P} c$ (“usual case”):

$$\dots \underline{bca} \dots \overset{\mathcal{P}}{\leftarrow\rightsquigarrow} \dots \underline{bac} \dots \quad \text{and} \quad \dots \underline{acb} \dots \overset{\mathcal{P}}{\leftarrow\rightsquigarrow} \dots \underline{cab} \dots .$$

The \mathcal{P} -Knuth equivalence is defined to be the closure of these moves.

Example: \mathcal{S}_3 

\mathcal{P} -Knuth moves and equivalence

For each equivalence class Γ , consider

$$\tilde{\gamma}_{\mathcal{P},\Gamma} = \sum_{w \in \Gamma} t^{|\text{f-inv}_{\mathcal{P}}(w)|} F_{\text{des}_{\mathcal{P}}(w)}$$

so that $\tilde{\gamma}_{\mathcal{P},S_n} = \sum_{\Gamma} \tilde{\gamma}_{\mathcal{P},\Gamma}$.

Question

Is $\tilde{\gamma}_{\mathcal{P},\Gamma}$ a symmetric function?

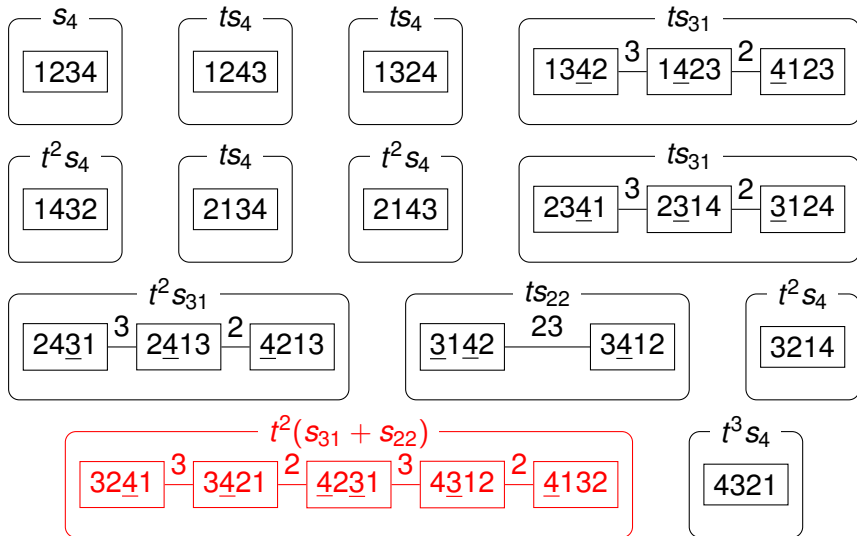
Question

Is $\tilde{\gamma}_{\mathcal{P},\Gamma}$ Schur-positive?

Question

Is $\tilde{\gamma}_{\mathcal{P},\Gamma}$ a single Schur function?

Example: $\mathcal{P} = \mathcal{P}_{(2,1),4}$



Main conjecture

Lemma

If w and w' are \mathcal{P} -Knuth equivalent, then $|\text{f-inv}_{\mathcal{P}}(w)| = |\text{f-inv}_{\mathcal{P}}(w')|$.

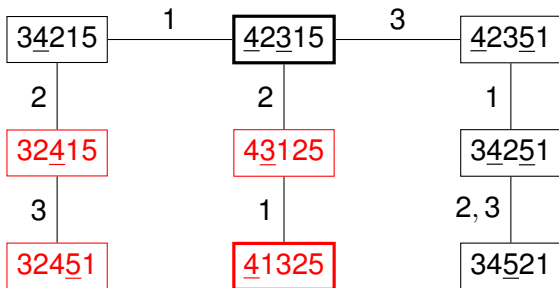
As a result, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^{|\text{f-inv}_{\mathcal{P}}(w)|} \cdot (\tilde{\gamma}_{\mathcal{P},\Gamma})_{t=1}$ for any $w \in \Gamma$.

Conjecture

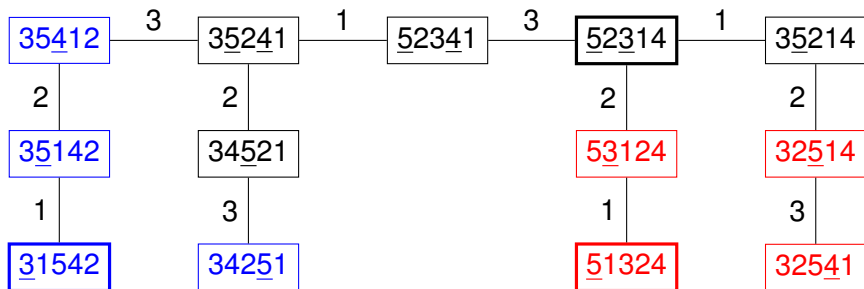
If \mathcal{P} is a natural unit interval order on $[1, n]$, then $(\tilde{\gamma}_{\mathcal{P},\Gamma})_{t=1}$ for each \mathcal{P} -Knuth equivalence class Γ is a Schur positive symmetric function.

This is a refinement of the theorem of Shareshian-Wachs.

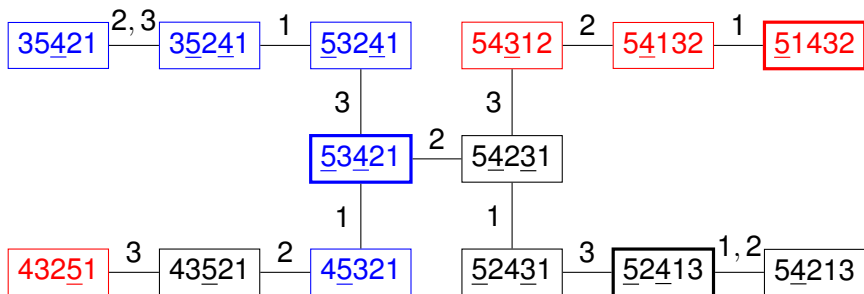
Example: $\mathcal{P} = \mathcal{P}_{(2,2,1),5}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^2(s_{32} + s_{41})$



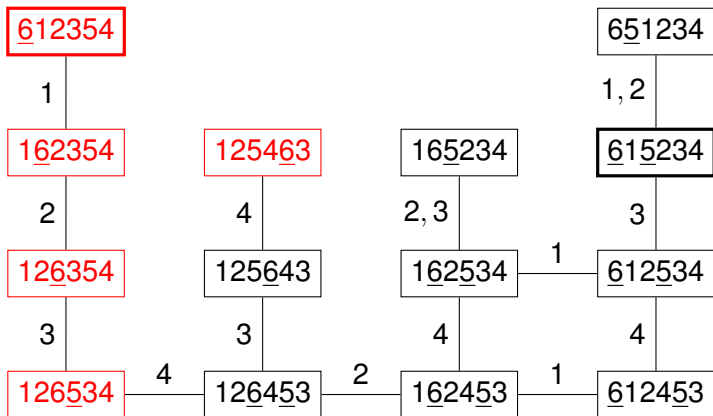
Example: $\mathcal{P} = \mathcal{P}_{(2,1,1),5}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^3(s_{32} + 2s_{41})$



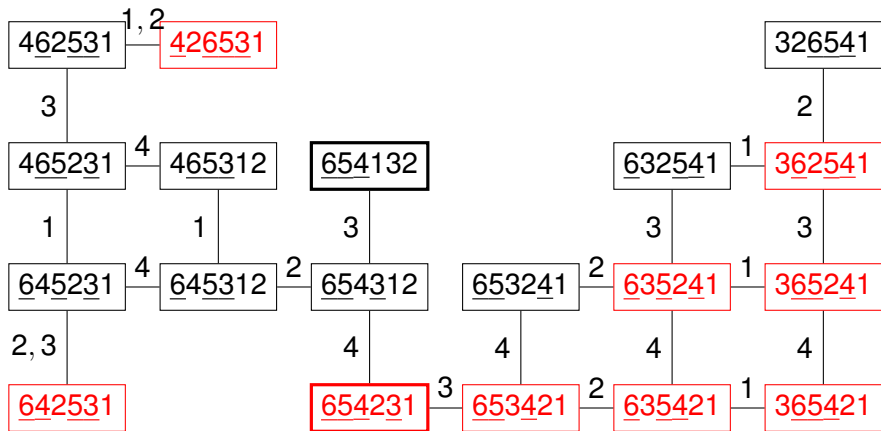
Example: $\mathcal{P} = \mathcal{P}_{(3,2,1),5}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^3(2s_{32} + s_{41})$



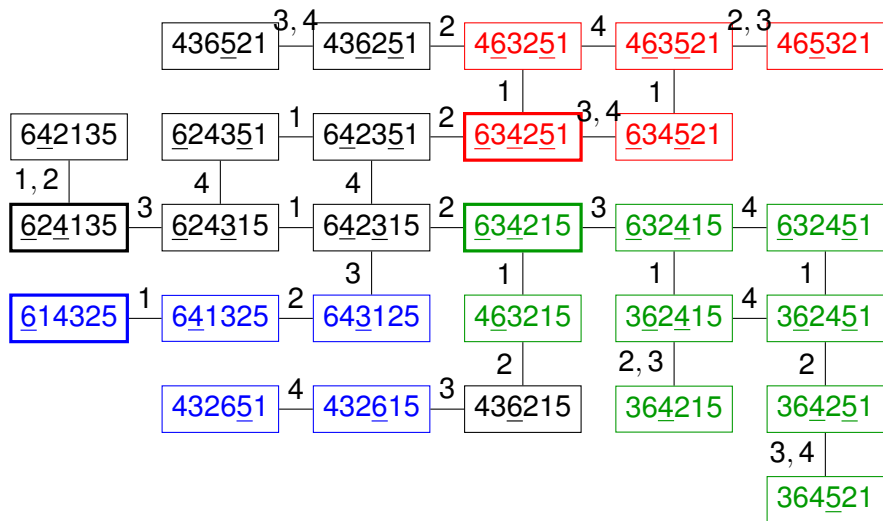
Example: $\mathcal{P} = \mathcal{P}_{(4,3,2,1),6}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^2(s_{42} + s_{51})$



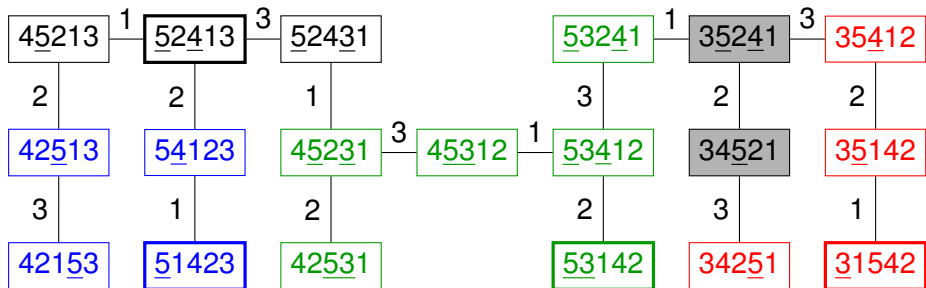
Example: $\mathcal{P} = \mathcal{P}_{(5,4,2,1),6}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^2(s_{2211} + s_{3111})$



Example: $\mathcal{P} = \mathcal{P}_{(3,3,2,1),6}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^4(s_{33} + 2s_{42} + s_{51})$



Example: $\mathcal{P} = \mathcal{P}_{(3,1,1),5}$, $\tilde{\gamma}_{\mathcal{P},\Gamma} = t^3(s_{311} + s_{32} + 2s_{41})$



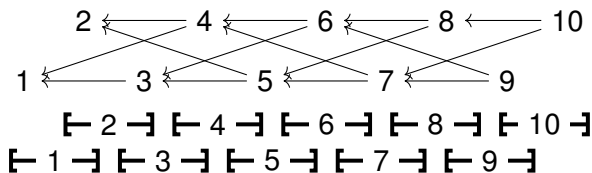
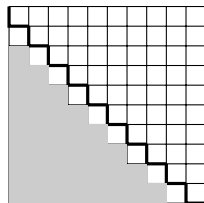
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Ladder order

Definition

We say that \mathcal{P} is a ladder order if it is isomorphic to $\mathcal{P}_{(n-2, n-3, \dots, 1), n}$ for some n .



Definition

We say that $X \subset [1, n]$ is a ladder in \mathcal{P} if $\mathcal{P}|_X$ is a ladder order.

For example, any $[a, b] \subset [1, n]$ is a ladder in $\mathcal{P}_{(n-2, n-3, \dots, 1), n}$.

Climbing a ladder

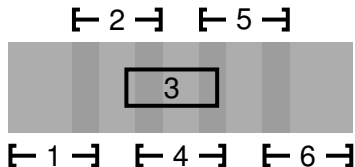
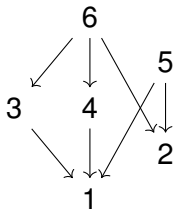
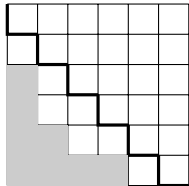
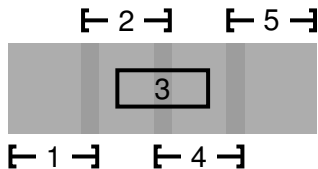
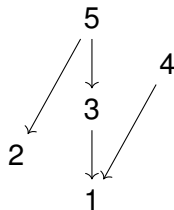
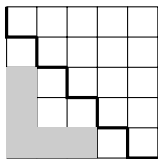
Definition

We say that someone is climbing a ladder in \mathcal{P} or \mathcal{P} is ladder-climbing if there exist $x, y_1, \dots, y_k \in [1, n]$ such that

- $x \notin \{y_1, y_2, \dots, y_k\}$,
- $\{y_1, y_2, \dots, y_k\}$ is a ladder in \mathcal{P} , and
- $y_1 \leftarrow_{\mathcal{P}} x \leftarrow_{\mathcal{P}} y_k$.

In such a case, we also say that x is climbing a ladder in \mathcal{P} or x is climbing (the ladder) $\{y_1, \dots, y_k\}$ in \mathcal{P} . Otherwise we say that no one is climbing a ladder in \mathcal{P} or \mathcal{P} is not ladder-climbing.

Example: $\mathcal{P}_{(3,1,1),5}$ and $\mathcal{P}_{(4,2,1,1),6}$



Theorem

\mathcal{P} is not ladder-climbing if and only if \mathcal{P} avoids $\mathcal{P}_{(3,1,1),5}$ and $\mathcal{P}_{(4,2,1,1),6}$.

Main theorem

Theorem (Main theorem)

Suppose that \mathcal{P} is a natural unit interval order that is not ladder-climbing. Then for any \mathcal{P} -Knuth equivalence class Γ , there exist $\lambda_1, \dots, \lambda_k$ such that

- 1 $\tilde{\gamma}_{\mathcal{P}, \Gamma} = t^{|\text{f-inv}_{\mathcal{P}}(\Gamma)|} (s_{\lambda_1} + \dots + s_{\lambda_k})$ where $|\text{f-inv}_{\mathcal{P}}(\Gamma)|$ is $|\text{f-inv}_{\mathcal{P}}(w)|$ for any $w \in \Gamma$, and
- 2 the lengths of $\lambda_1, \dots, \lambda_k$ are all equal.

It is likely that the first part of the main theorem is valid for any natural unit interval order \mathcal{P} albeit the second part is not true.

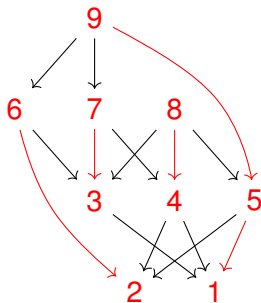
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\mathcal{P} -tableaux

A tableau T is said to be a \mathcal{P} -tableau if

- the entries of T are exactly $1, 2, \dots, n$ without repetition,
- the entries are increasing along columns with respect to \mathcal{P} and
- the entries are nondecreasing along rows with respect to \mathcal{P} .



1	4	3	2
5	8	7	6
9			

Properties of \mathcal{P} -Robinson-Schensted algorithm

Theorem

Suppose that \mathcal{P} is not ladder-climbing. Then there exists a bijection $\mathcal{P}\text{-RS} : \mathcal{S}_n \rightarrow \bigsqcup_{\lambda \vdash n} \mathcal{P}\text{-Tab}_\lambda \times \text{SYT}_\lambda$ such that:

- (A) $\text{des}_{\mathcal{P}}(w) = \{n - x \mid x \in \text{des}(Q)\} = \text{des}(\text{evac}(Q))$.
- (B) $w \sim_{\mathcal{P}} \text{read}(P)$ (reading word of P).
- (C) If $w' \in \mathcal{S}_n$ satisfies $\mathcal{P}\text{-RS}(w') = (P', Q')$ and $w \sim_{\mathcal{P}} w'$, then the first columns of P and P' are of the same length.

Unlike the usual Robinson-Schensted algorithm, in general we do not have the same \mathcal{P} -tableaux for w, w' such that $w \sim_{\mathcal{P}} w'$.

Proof of the main theorem

Theorem (Main theorem)

Suppose that \mathcal{P} is not ladder-climbing. Then for any \mathcal{P} -Knuth equivalence class Γ , there exist $\lambda_1, \dots, \lambda_k$ such that

- 1 $\tilde{\gamma}_{\mathcal{P}, \Gamma} = t^{|\text{f-inv}_{\mathcal{P}}(\Gamma)|} (s_{\lambda_1} + \dots + s_{\lambda_k})$ where $|\text{f-inv}_{\mathcal{P}}(\Gamma)|$ is $|\text{f-inv}_{\mathcal{P}}(w)|$ for any $w \in \Gamma$, and
- 2 the lengths of $\lambda_1, \dots, \lambda_k$ are all equal.

Sketch of the proof.

Using (B) one can show that \mathcal{P} -RS restricts to a bijection $\Gamma \simeq \bigsqcup_{i=1}^k \{P_i\} \times \text{SYT}_{\text{sh}(P_i)}$ for certain \mathcal{P} -tableaux P_1, \dots, P_k . Now by (A), it follows that $\tilde{\gamma}_{\mathcal{P}, \Gamma} = t^{|\text{f-inv}_{\mathcal{P}}(\Gamma)|} \sum_{i=1}^n s_{\text{sh}(P_i)}$, which proves (1). The first columns of P_1, \dots, P_k are of the same length by (C), and thus (2) follows. □

Thank you!