Nilpotent Higgs bundles and the Hodge metric on the Calabi-Yau moduli

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We study an algebraic function on orbits of nilpotent matrices and show how it gives geometric applications by relating the algebraic function with the curvature formula on homogeneous spaces.

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Partitions and nilpotent matrices

- A **partition** of n is a non-increasing array (λ_i) of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ satisfying $\sum_{p=1}^{n} \lambda_p = n$.
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- The space $\mathcal{P}_n$ has a natural partial ordering:

  λ is said to **dominate** μ $(\lambda \geq \mu)$ if for all $p \leq n$, $\sum_{i=1}^{p} \lambda_i \geq \sum_{i=1}^{p} \mu_i$.

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  For example, $(4) > (3,1) > (2,2) > (2,1,1) > (1,1,1,1)$.
- Given a partition $\lambda \in \mathcal{P}(n)$, define $X^\lambda = diag(J_{\lambda_1}, \cdots, J_{\lambda_k})$, where J_i is an elementary Jordan block of type i:

$$J_i = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ r_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & r_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & r_{i-1} & 0 \end{pmatrix}, \quad r_p = \sqrt{p(i-p)}.$$

Consider the function $K : sl(n, \mathbb{C}) \to \mathbb{R}$ given by

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Lemma (Ness 84', Schmid-Vilonen 99')

Let A be a nilpotent matrix in $sl(n, \mathbb{C})$. Then A is a critical point of the function $K(A)$ on its adjoint orbit $\mathcal{O}_A$ if and only if A is unitarily conjugate to $c \cdot X^\lambda$ for $c \in \mathbb{C}^*$.
Moreover, the function K on $\mathcal{O}_A$ assumes its minimum exactly on the critical set.

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It is a theorem in geometric invariant theory.
In a joint work with Dai, we prove a generalized theorem and give an independent proof of the lemma as a byproduct.

For each $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathcal{P}_n$, we associate a constant

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Let A be a nilpotent matrix in $sl(n, \mathbb{C})$, we say it is of Jordan type at most $\lambda \in \mathcal{P}_n$ if the block sizes of A's Jordan normal form give the partition μ where $\mu \leq \lambda$.

Proposition

Suppose $A \in \mathcal{N}$ is of Jordan type at most $\lambda \in \mathcal{P}_n$, then

$$K(A) \geq C_\lambda.$$

Equality holds if and only if A is $SU(n)$-conjugate to $c \cdot X^\lambda$, for some constant $c \in \mathbb{C}^*$.

Young diagram and the conjugate partition

- Given a partition $\lambda \in \mathcal{P}(n)$, define a new partition $\lambda^t = (\lambda^t_1, \cdots, \lambda^t_n) \in \mathcal{P}$ where $\lambda^t_j = |\{i|\lambda_i \geq j\}|$, called the **conjugate partition** of λ.

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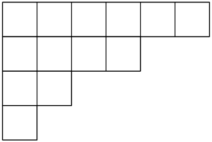
The Young diagram helps us see the conjugate partition more explicitly. For $\lambda \in \mathcal{P}(n)$, form k rows of empty boxes such that the ith row has λ_i boxes. Such array is called the **Young diagram** of λ.

Partition λ: $(6, 4, 2, 1)$

Young Diagram:



Conjugate Partition λ^t: $(4, 3, 2, 2, 1, 1)$

Compositions and conjugate partitions

- A **composition** of n is an array (r_i) of positive integers $r_1, \cdots, r_m$ satisfying $\sum_{i=1}^{m} r_i = n$. Let $\mathcal{C}_n$:=the space of compositions of n.
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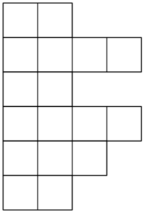
- r_i's are not necessarily in non-increasing order.
- For a composition $\mathcal{R} \in \mathcal{C}_n$, we can also define a conjugate partition of $\mathcal{R}$: form m rows of empty boxes such that the ith row has r_i boxes, we obtain **an analogue of Young diagram**.

Composition $\mathcal{R}$: $(2, 4, 2, 4, 3, 2)$

Generalized Young Diagram:



Conjugate Set Partition: $(\{6\}, \{6\}, \{1, 2\}, \{1, 1\})$

Conjugate Partition $\mathcal{R}^t$: $(6, 6, 2, 1, 1, 1)$

Definition

A matrix A is said to be of **type** $\mathcal{R} \in \mathcal{C}_n$ *if A is conjugate to*

$$\begin{pmatrix} 0 & & & & \\ A_1 & 0 & & & \\ & A_2 & 0 & & \\ & & \ddots & \ddots & \\ & & & A_{m-1} & 0 \end{pmatrix}, \tag{1}$$

where each A_i is a $r_i \times r_{i+1}$ matrix, for $1 \leq i \leq m-1$.

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Lemma

For a matrix A of type $\mathcal{R} \in \mathcal{C}_n$, A is of Jordan type at most $\mathcal{R}^t$ and it is sharp. The function satisfies

$$K(A) \geq C_{\mathcal{R}^t}. \tag{2}$$

How to make use of such algebraic inequalities into geometry (Higgs bundles)?

How to make use of such algebraic inequalities into geometry (Higgs bundles)?

This work is inspired by Xu Wang's paper: Curvature restrictions on a manifold with a flat Higgs bundle, arXiv 1608.00777v2

The philosophy is that we already know a lot information from the algebraic structure of nilpotent Higgs bundle without solving equations.

Higgs bundles

Definition

A Higgs bundle over a complex manifold M consists of a pair (E, ϕ):

- *E is a holomorphic vector bundle over M;*
- *$\phi \in \Omega^1(End(E))$ satisfying the integrability condition $\phi \wedge \phi = 0$.*

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A Hermitian metric h on a degree 0 Higgs bundle (E, ϕ) is called harmonic if it satisfies the Hitchin equation

$$F(D^h) + [\phi, \phi^{*_h}] = 0,$$

where

- D^h is the Chern connection on E uniquely determined by the holomorphic structure and the metric h,
- $F(D^h)$ is the curvature of D^h,
- ϕ^{*_h} is the adjoint of ϕ with respect to h.

- The harmonic metric h gives a Kähler metric g_M on M:

$$g_M\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) = tr(\phi_j \phi_k^{*_h}), \quad \phi_j = \phi\left(\frac{\partial}{\partial z_j}\right).$$

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The metric g_M is called Hodge metric on M.

The following proposition is the key link to the algebraic function K.

Proposition

Let (E, ϕ) be a degree 0 Higgs bundle over M which admits a harmonic metric h. Then away from zeros of ϕ_j, the holomorphic sectional curvature κ_j of g_M on the tangent plane $span_{\mathbb{C}}\{\frac{\partial}{\partial z_j}\}$ is

$$\kappa_j \leq -\frac{||[\phi_j, \phi_j^{*h}]||^2}{||\phi_j||^4}.$$

Proposition (Li 20')

Suppose at any point p such that ϕ_j is nilpotent of Jordan type at most $\lambda \in \mathcal{P}_n$, then the holomorphic sectional curvature $k_j(p)$ of the Hodge metric over M is bounded from above by $-C_\lambda$.

This is a pointwise estimate.

Riemann surface

Let $\Sigma = (S, J)$ be a compact Riemann surface of genus ≥ 2. The nonabelian Hodge correspondence (NAH) is a homeomorphism:

$$Hom^{+}(\pi_1(S), SL(n, \mathbb{C}))/SL(n, \mathbb{C}) \cong \mathcal{M}_{Higgs}(SL(n, \mathbb{C})),$$

where $\mathcal{M}_{Higgs}(G)$ consists of gauge equivalence classes of polystable G-Higgs bundles over Σ.

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- polystability is equivalent to existence of harmonic metric.
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In this case, we can apply the estimate on the holomorphic sectional curvature of the Hodge metric to obtain information of the NAH.

- Stratify the nilpotent cone of $\mathcal{M}_{Higgs}(G)$ according to Jordan types. For a nilpotent Higgs bundle (E, ϕ) over Σ, one can define its Jordan type $J(E, \phi) \in \mathcal{P}_n$:

$$J(E, \phi) = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathcal{P}_n,$$

where F_i is a holomorphic subbundle of E generated by $\ker(\phi^i)$ and $rank(F_i) - rank(F_{i+1}) = \lambda_i + \cdots + \lambda_n$.

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- Given a partition $\lambda \in \mathcal{P}_n$, using the unique irreducible representation $\tau_r : SL(2, \mathbb{C}) \to SL(r, \mathbb{C})$, one can define a natural representation $\tau_\lambda = \mathrm{diag}(\tau_{\lambda_1}, \cdots, \tau_{\lambda_n}) : SL(2, \mathbb{C}) \to SL(n, \mathbb{C})$.

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- The translation length of γ with respect to a representation $\rho : \pi_1(S) \to SL(n, \mathbb{C})$ is defined by

$$l_\rho(\gamma) := \inf_{x \in SL(n,\mathbb{C})/SU(n)} d(x, \rho(\gamma)x),$$

where $d(\cdot, \cdot)$ is the distance induced by the Riemannian metric.

For a Riemann surface structure Σ on S, the uniformization theorem gives rises to a representation $j_\Sigma : \pi_1(S) \to SL(2,\mathbb{R})$.

Theorem (Li 20')

Suppose a nilpotent polystable $SL(n,\mathbb{C})$-Higgs bundle (E,ϕ) over Σ is of Jordan type at most $\lambda \in P_n$. Let $\rho : \pi_1(S) \to SL(n,\mathbb{C})$ be its associated representation. Then there exists a positive constant $\alpha < 1$ such that

$$l_\rho \leq \alpha \cdot l_{\tau_\lambda \circ j_\Sigma},$$

unless $\mathbb{P}(\rho) = \mathbb{P}(\tau_\lambda \circ j_\Sigma)$.

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Idea: From the curvature estimate of the pullback metric of the harmonic map, we obtain the comparison between the pullback metric with the hyperbolic metric. Then we translate the metric comparison into the comparison between length spectrum. The rigidity also takes some work.

Note that (n) is maximal in $\mathcal{P}_n$. As a direct corollary,

Corollary

For any nilpotent polystable $SL(n,\mathbb{C})$-Higgs bundle over Σ, the associated representation ρ satisifies $l_\rho \leq \alpha \cdot l_{\tau_n \circ j_\Sigma}$ for some positive constant $\alpha < 1$, unless $\mathbb{P}(\rho) = \mathbb{P}(\tau_n \circ j_\Sigma)$.
As a result, the entropy of ρ satisfies if it is finite, then $h(\rho) \geq \sqrt{\frac{6}{n(n^2-1)}}$ and equality holds if and only if $\mathbb{P}(\rho) = \mathbb{P}(\tau_n \circ j_\Sigma)$.

- The entropy of a representation $\rho : \pi_1(S) \to SL(n,\mathbb{C})$ is defined as

$$h(\rho) := \limsup_{R\to\infty} \frac{\log(\#\{\gamma \in \pi_1(\Sigma) | l_\rho(\gamma) \leq R\})}{R}.$$

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- Potrie and Sambarino 17' showed for any Hitchin representation ρ, $h(\rho) \leq \sqrt{\frac{6}{n(n^2-1)}}$ (with an appropriate normalization) and the equality holds only if $\mathbb{P}(\rho) = \mathbb{P}(\tau_n \circ j_\Sigma)$ for some Riemann surface Σ. We can see that the nilpotent cone possesses an opposite behavior of the Hitchin section.

Period Domain

- Let X be a compact Kähler manifold of dimension n. A $(1,1)$-form ω is called a polarization of X if $[\omega]$ is the first Chern class of an ample line bundle over X. Using the form ω, one can define the k-th primitive cohomology $P^k(X,\mathbb{C}) \subset H^k(X,\mathbb{C})$.

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- Let $H^{p,q} = P^k(X,\mathbb{C}) \cap H^{p,q}(X)$ for $0 \leq p, q \leq k$. Then we have $H = P^k(X,\mathbb{C}) = \Sigma_p H^{p,q}$ and $H^{p,q} = \overline{H^{q,p}}$. We call $\{H^{p,q}\}$ the Hodge decomposition of H.

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- Q is a nondegenerate quadratic form on H:

$$Q(\phi,\psi) = (-1)^{\frac{k(k-1)}{2}} \int_X \phi \wedge \psi \wedge \omega^{n-k}, \quad \phi, \psi \in H$$

  satisfying the two Hodge-Riemann relations:
  (*) $Q(H^{p,q}, H^{p',q'}) = 0$ unless $p' = n-p, q' = n-q$;
  (**) $b(\cdot,\cdot) = Q(i^{p-q}\cdot, \bar{\cdot})$ is a Hermitian inner product on $H^{p,k-p}$ for each p.

Period Domain

The space $\mathcal{D} = \mathcal{D}(H, Q, k, \{h^{p,q}\})$ consisting of all Hodge structures of weight k with fixed dimension $h^{p,q}$ of $H^{p,q}$, polarized by Q, is called the period domain.

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- Let U be an open neighborhood of the universal deformation space of X. Assume that U is smooth. A polarized variation of Hodge structures is equivalent to the map

$$\mathcal{P} : U \to \mathcal{D}, \quad X' \to \{P^k(X', \mathbb{C}) \cap H^{p,q}(X')\}_{p+q=k},$$

called Griffiths' period map.

Period Map

- (Griffiths 68') The period map is holomorphic and its tangential map has image in the horizontal distribution $T^h\mathcal{D}$:
  tangent vector is of type $\mathcal{R} = (h^{k,0}, h^{k-1,1}, \cdots, h^{0,k})$.

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- The period domain $\mathcal{D}$ can be written as $\mathcal{D} = G/V$ equipped with a G-invariant Hermitian metric h induced by trace form.
  e.g. $k = 2m+1, G = Sp(n, \mathbb{R}), \dim H = 2n, V = \prod_{p \leq m} U(h^{p,q})$.

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- (Griffiths-Schmid 69') The horizontal distribution $T^h\mathcal{D}$ always has negative holomorphic sectional curvature.

Here we give an effective estimate of the holomorphic sectional curvature of $T^h\mathcal{D}$ by comparing the curvature formula on D and the function K.

Theorem (Li 20')

The G-invariant Hermitian metric h on $\mathcal{D} = \mathcal{D}(H, Q, k, \{h^{p,q}\})$ has holomorphic sectional curvature in the direction $\xi \in T^h\mathcal{D}$ satisfying

$$K(\xi) \leq -C_{\mathcal{R}^t}.$$

Moreover, the equality can be achieved in some direction ξ.

Calabi-Yau moduli

In particular, we can apply the result directly to the Hodge metric on the Calabi-Yau moduli spaces.

- A polarized Calabi-Yau m-manifold is a pair (X, ω) of a compact algebraic manifold X of dimension m with vanishing first Chern class and a Kähler form $\omega \in H^2(X, \mathbb{Z})$.

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- (Tian) The universal deformation space $\mathcal{M}_X$ of polarized Calabi-Yau m-manifolds is smooth.
- The tangent space $T_{X'}\mathcal{M}_X$ of $\mathcal{M}_X$ at X' can be identified with $H^1(X', T_{X'})$. Let $n = dim\mathcal{M}_X$.
- $h^{p,m-p}$:=the dimension of the $(p, m-p)$-primitive cohomology group of (X, ω). So $h^{m,0} = h^{0,m} = 1, h^{m-1,1} = h^{1,m-1} = n$.

We have two natural metrics on $\mathcal{M}_X$:

- The Hodge metric ω_H on $\mathcal{M}_X$ was first defined in Lu as the pullback metric $\mathcal{P}^*h$ on $\mathcal{M}_X$ by the period map $\mathcal{P} : \mathcal{M}_X \to \Gamma\backslash\mathcal{D}$.

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- Let $\mathcal{F}^m$ be the first Hodge bundle over $\mathcal{M}_X$ formed by $H^{m,0}(X)$, the Weil-Petersson metric on $\mathcal{M}_X$ is defined as $\omega_{WP} = c_1(F^m)$.

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The Weil-Petersson metric and the Hodge metric on $\mathcal{M}_X$ are closely related.

Proposition

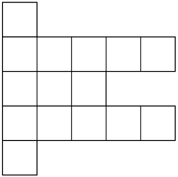
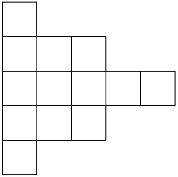
(1) In the case of twofold, $\omega_H = 2\omega_{WP}$;
(2) (Lu 01') In the case of threefold, $\omega_H = (n+3)\omega_{WP} + Ric(\omega_{WP})$;
(3) (Lu-Sun 15') In the case of fourfold, $\omega_H = 2(n+2)\omega_{WP} + 2Ric(\omega_{WP})$;
(4) (Lu-Sun 15' 16') In the case of higher dimension, we only have the inequality $\omega_H \geq 2(n+2)\omega_{WP} + 2Ric(\omega_{WP}) \geq 2\omega_{WP}$.

An important application is the following estimate of the holomorphic sectional curvatures of the Hodge metric on $\mathcal{M}_X$.

Theorem (Li 20')

For a polarized Calabi-Yau m-manifold (X, ω) of Hodge type $\mathcal{R}$, let n be the dimension of the universal deformation space $\mathcal{M}_X$. Then the Hodge metric over $\mathcal{M}_X$ has its holomorphic sectional curvature bounded from above by a negative constant $c_m = -C_{\mathcal{R}^t}$.
In particular,
(1) $c_3 = -\frac{2}{n+9}$.
(2) $c_4 = -\frac{1}{2(\min\{a,n\}+4)}$ *for* $a = h^{2,2}$.
(3) $c_5 = -\frac{2}{(9\min\{a,n\}+a+25)}$ *for* $a = h^{3,2}$.

For example, prove for $n = 4$.
Consider a matrix of type $\mathcal{R} = (1, n, a, n, 1)$, where $a = h^{2,2}$.
In case $a \leq n$, the conjugate partition $\mathcal{R}^t$ is $\lambda_1 = (5, 3^{a-1}, 1^{2n-2a})$.
In case $a \geq n$, the conjugate partition $\mathcal{R}^t$ is $\lambda_2 = (5, 3^{n-1}, 1^{a-n})$.

Composition $\mathcal{R}$:	$(1, n = 5, a = 3, n = 5, 1)$	$(1, n = 3, a = 5, n = 3, 1)$
Generalized Young Diagram:		
Conjugate Set Partition:	$(\{5\}, \{3\}, \{3\}, \{1, 1\}, \{1, 1\})$	$(\{5\}, \{3\}, \{3\}, \{1\}, \{1\})$
Conjugate Partition $\mathcal{R}^t$:	$(5, 3^2 = 3^{a-1}, 1^4 = 1^{2n-2a})$	$(5, 3^2 = 3^{n-1}, 1^3 = 1^{a-n})$

$$C_{\lambda_1} = \frac{12}{5(5^2-1)+(a-1)3(3^2-1)} = \frac{1}{10+2(a-1)} = \frac{1}{2(a+4)},$$

$$C_{\lambda_2} = \frac{12}{5(5^2-1)+(n-1)3(3^2-1)} = \frac{1}{10+2(n-1)} = \frac{1}{2(n+4)}.$$

Then $c_4 = -\frac{1}{2(\min\{a,n\}+4)}$ for $a = h^{2,2}$.

Remark

- *For* $m = 3$*, Lu 01' gave the upper bound* $-\frac{1}{(\sqrt{n}+1)^2+1}$*. Here we improve to the upper bound* $-\frac{2}{n+9}$*.*
- *For* $m = 4$*, Lu-Sun 04' showed the upper bound is* $-\frac{1}{2(n+4)}$*. Here we obtain a refined upper bound by replacing* n *by* $\min\{a, n\}$ *where* $a = h^{2,2}$*.*
- *For* $m = 5$ *and higher, our estimates are new.*
- *By the Schwarz-Yau lemma, a refined bound of holomorphic sectional curvatures will give a refined estimates of the Weil-Petersson metric on a complete algebraic curve inside the moduli space* $\mathcal{M}_X$*. As additional applications, such estimates can give refined Arakelov-type inequalities.*

Thank You.