Dominating surface-group representations into $\mathrm{PSL}_2(\mathbb{C})$ in the relative representation variety

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1. Surface-group representations into $\mathrm{PSL}_2(\mathbb{C})$

$S_{g,k}$: oriented surface of genus $g \geq 0$ with $k \geq 1$ punctures, labelled by $p_1, \cdots, p_k$.

(We assume that $2g - 2 + k > 0$.)

Π: the fundamental group of $S_{g,k}$.

$\rho : \Pi \to \mathrm{PSL}_2(\mathbb{C})$: non-Fuchsian representation.

Theorem (Gupta-Su, arXiv:2003.13572)

There exists a Fuchsian representation that strictly dominates ρ in the simple length-spectra and preserves the boundary lengths.

For a fixed k -tuple $\mathcal{L} = (l_1, l_2, \ldots, l_k) \in \mathbb{R}^k_{\geq 0}$, the *relative representation variety* for the surface-group Π is

$$\mathrm{Hom}(\Pi, \mathcal{L}) = \{\rho : \Pi \to \mathrm{PSL}_2(\mathbb{C}) \mid l_\rho(\gamma_i) = l_i\}$$

where γ_i is the loop around p_i.

A Fuchsian representation $j \in \mathrm{Hom}(\Pi, \mathcal{L})$ is said to strictly dominate a representation $\rho \in \mathrm{Hom}(\Pi, \mathcal{L})$ if

$$\sup_\gamma \frac{l_\rho(\gamma)}{l_j(\gamma)} < 1$$

where γ varies over all non-peripheral essential simple closed curves on $S_{g,k}$.

Remarks:

- (Thurston) A Fuchsian representation cannot have a strictly dominating Fuchsian representation in the same relative representation variety.
- (Gueritaud-Kassel-Wolff) Closed surface-group representations into $\mathrm{PSL}_2(\mathbb{R})$, using "unfolding" contruction.
- (Deroin-Tholozan) More general domination result for representations of a closed surface-group into the isometry group of smooth Riemannian $\mathrm{CAT}(-1)$ spaces, using the theory of harmonic maps.

Our theorem is independently proved by Sagman using harmonic maps.

Our proof avoids harmonic maps, which relies instead on the pleated-surface interpretation of the Fock-Goncharov coordinates of a framed representation into $\mathrm{PSL}_2(\mathbb{C})$, as exploited in the work of Gupta-Mj and Gupta.

The idea is to straightening the pleated plane in $\mathbb{H}^3$ determined by the Fock-Goncharov coordinates of a framed representation, and applying strip-deformations.
(Thurston, Papadopoulos-Theret, Gueritaud-Kassel-Wolff, Danciger-Gueritaud-Kassel)

Corollary

Let $\rho \in \mathrm{Hom}(\Pi, \mathcal{L})$ be a representation such that the length of each peripheral curve is bounded above by L. Then there exists a pants decomposition of $S_{g,k}$ such that the ρ -lengths of the pants curves are less than the Bers constant $B(g, k, L)$.

The above result improves on a result of Whang.

Conjecture

A generic representation $\rho : \Pi \to \mathrm{PSL}_n(\mathbb{C})$ has a strictly dominating Hitchin representation $\Pi \to \mathrm{PSL}_n(\mathbb{R})$ in the same relative representation variety.

A closely related "Metric Domination Conjecture" in the case of a closed surface is proposed by Dai-Li.

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2. Framed representations and pleated laminations

Let $\rho \in \mathrm{Hom}(\Pi, \mathcal{L})$ is a non-Fuchsian representation.

Definition. A representation $\rho : \Pi \to \mathrm{PSL}_2(\mathbb{C})$ is said to be degenerate if either

(a) the image of ρ has a global fixed point on $\mathbb{CP}^1$, and $\rho(\gamma_i)$ is parabolic or identity for each peripheral loop γ_i, or

(b) the image of ρ preserves a two-point set on $\mathbb{CP}^1$, which is fixed by each $\rho(\gamma_i)$ (where $1 \leq i \leq k$).

A representation is then said to be non-degenerate if it is not degenerate. In this talk we shall assume that ρ is non-degenerate.

Given a non-degenerate representation ρ, one can construct a non-degenerate framing β by assigning to each puncture one of the fixed points of the holonmy (monodromy) around it.

To define β, let us fix a finite-area hyperbolic metric on $S_{g,k}$ such that the punctures are cusps. Passing to the universal cover, the Farey set F_∞ is the points in the ideal boundary that are the lifts of the punctures.

Note that F_∞ is equipped with an action of the surface-group Π. A ρ-equivariant map $\beta : F_\infty \to \mathbb{C}P^1$ is called a frame. The pair (ρ, β) is called a framed representation.

Definition. A framed representation (ρ, β) is said to be degenerate if either of the following conditions hold:

(1) The image of the map β is a single point $p \in \mathbb{C}P^1$, the monodromy around each puncture is parabolic with fixed point p or the identity element, and $\rho(\gamma)$ fixes p for each $\gamma \in \Pi$, or

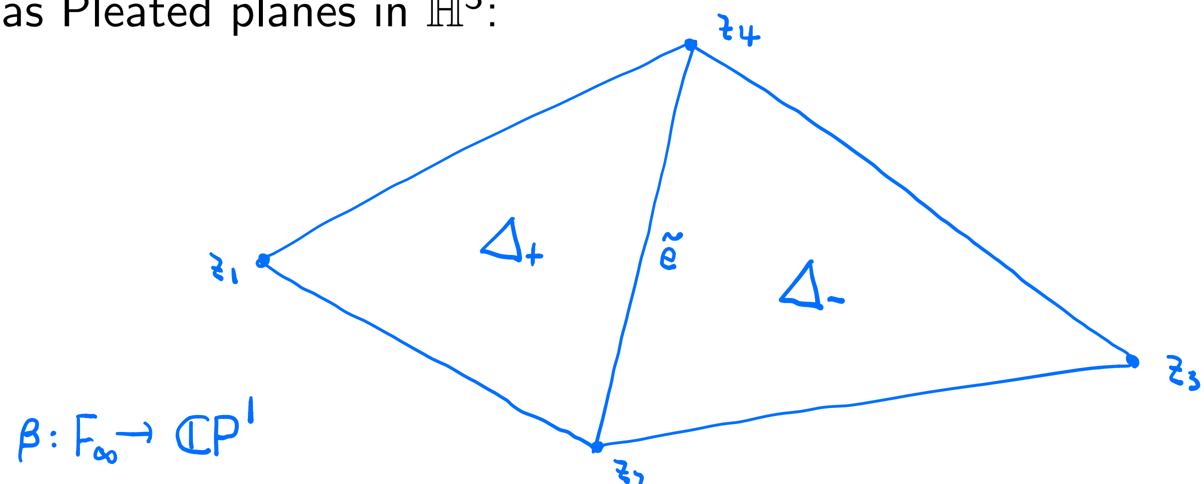
(2) The image of the map β is a pair of points $\{p-, p+\} \in \mathbb{C}P^1$, that is fixed by the monodromy around each puncture, and preserved (i.e. fixed or permuted) by $\rho(\gamma)$ for each $\gamma \in \Pi$.

Given a non-degenerate representation ρ, one can construct a non-degenerate framing β by assigning to each puncture one of the fixed points of the holonmy/monodromy around it.

Fock-Goncharov coordinates: cross-ratios

Theorem (Allegretti-Bridgeland). For a non-degenerate framed representation (ρ, β), there is an ideal triangulation T such that the Fock-Goncharov coordinates for (ρ, β) are well-defined and non-zero.

A geometric interpretation of the Fock-Goncharov coordinates as Pleated planes in $\mathbb{H}^3$:



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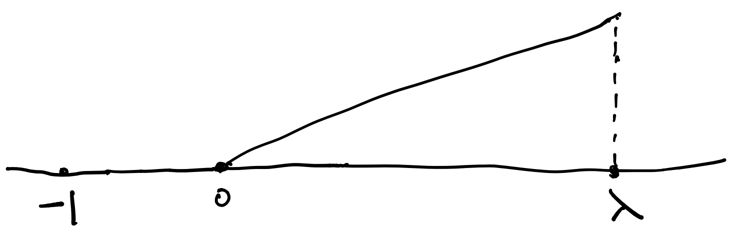
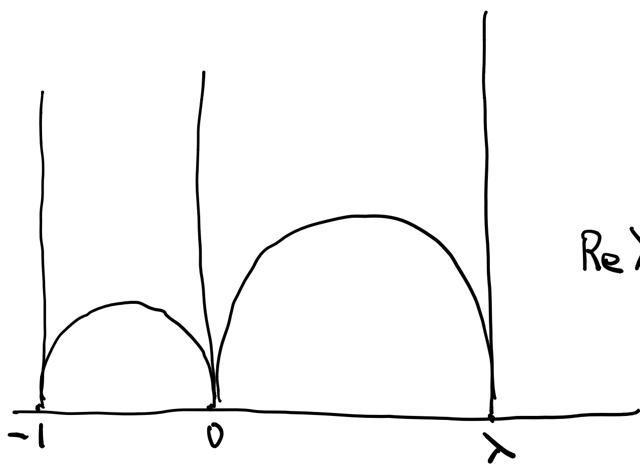
$$(-1, 0, \lambda, \infty) = \frac{\lambda - 0}{-1 - 0}\,\frac{-1-\infty}{\lambda - \infty} = -\lambda$$

Im λ : bending



Re λ: shearing



3. Proof of the theorem

(The work of Gupta:)

- (ρ, β): (nondegenerate) framed representation.
- T: ideal triangulation.
- $\{c(e)\}_{e \in T} \in (\mathbb{C}^*)^{|T|}$: Fock-Goncharov coordinates.
- $\Psi : \widetilde{S} \to \mathbb{H}^3$: ρ-equivariant pleated plane.
- $\widehat{\Psi} : \widetilde{S} \to \mathbb{H}^3$: straightening of Ψ.
- $j_0 : \Pi \to \mathrm{PSL}_2(\mathbb{R})$: Fuchsian representation induced by $\widehat{\Psi}$.
- $\widehat{S} = \mathbb{H}^2 / j_0(\Pi)$
- λ: pleated measured lamination on $\widehat{S}$. Each geodesic boundary component of $\widehat{S}$ has at least one leaf of λ spiralling onto it.

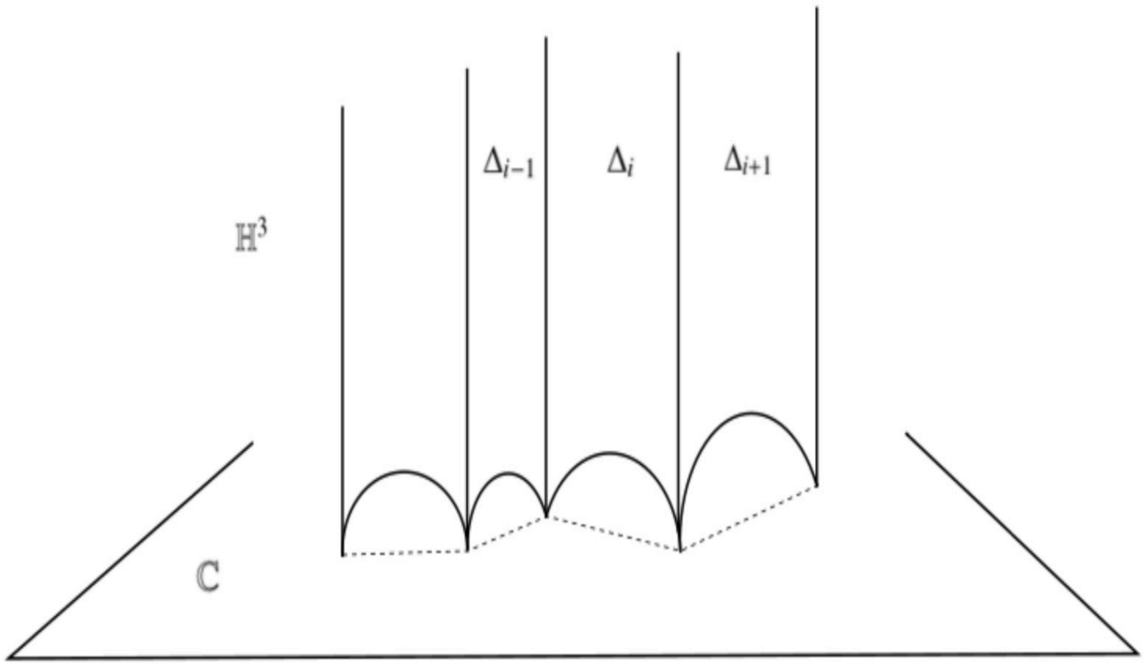
Our first observation is:

Lemma

The j_0-length of the boundary curve around the i-th puncture p_i is equal to l_i, for $1 \leq i \leq k$. That is, $j_0 \in \mathrm{Hom}(\Pi, \mathcal{L})$ as well.

This implies that the Fuchsian representation j_0 weakly dominate ρ:

$$\sup_{\gamma} \frac{l_\rho(\gamma)}{l_{j_0}(\gamma)} \leq 1$$



$$\hat{\Delta}_i \longrightarrow \hat{\Delta}_{i+1}$$

$$z \mapsto |s_i| z + c_i, \quad c_i \in \mathbb{R}$$

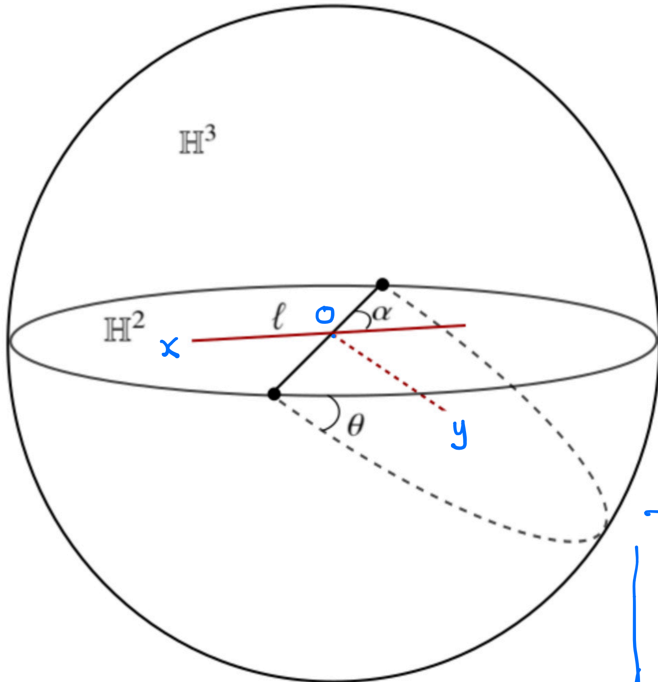
monodromy: $z \mapsto |s_0 s_1 \cdots s_{n-1}| z + c, \quad c \in \mathbb{R}.$

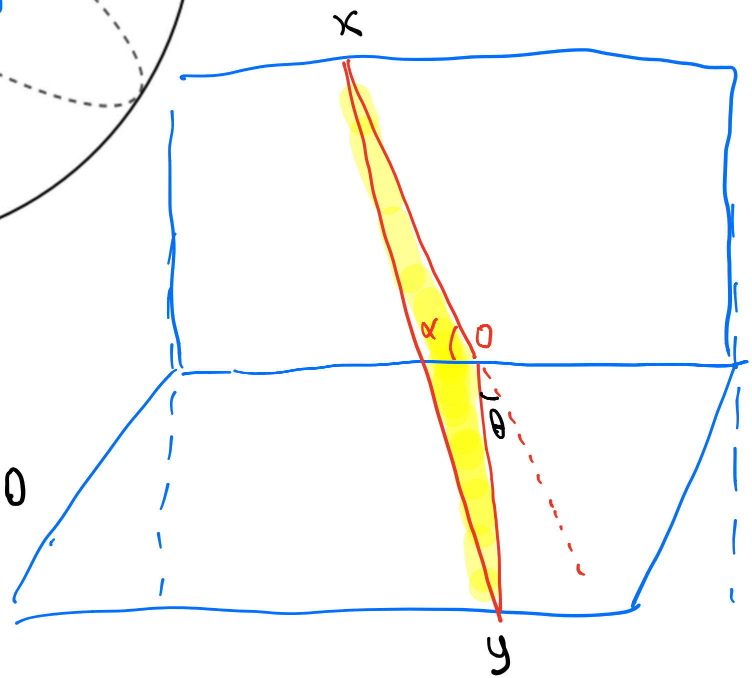
A geometric lemma to quantify how the translation length of a non-peripheral loop changes under pleating:

Lemma

For any $L > 0, \alpha \in (0, \pi/2)$ and $\theta \in (-\pi, \pi)$, there is a constant $C > 0$ such that the following holds:

Let $\mathbb{H}^2$ be isometrically embedded as the equatorial plane in $\mathbb{H}^3$, containing a geodesic segment ℓ and a bi-infinite geodesic line γ, such that the two intersect at an angle at least α, and ℓ has length at least L on either side of γ. Let $\hat{\ell}$ be the piecewise-geodesic in $\mathbb{H}^3$ obtained when the equatorial plane is pleated along γ by a pleating angle at least θ. Then the distance in $\mathbb{H}^3$ between the endpoints of $\hat{\ell}$ is less than $|\ell| - C$.





$$1+\cos\beta > \delta = \delta(\alpha,\theta) > 0$$

Lemma (Wolpert)

Given a hyperbolic surface X of finite type, with finitely many geodesic boundaries and cusps, there exists a $D > 0$ such that any non-peripheral simple closed geodesic γ on X remains at least distance D away from the geodesic boundary components, and the standard horoball neighborhoods of the cusps.

Let γ be any simple closed geodesic on S. We can decompose γ into a finite union of geodesic arcs $\{\gamma_j\}$ such that each γ_j has endpoints on λ, and has its interior disjoint from λ. Since γ does not cross some collar neighborhood of ∂S and a horodisk-neighborhoods around the cusps. As a result, γ_j satisfies the hypotheses of Lemma for some L, a and θ (which are all independent of the choice of γ). Note that the length of γ_j is uniformly comparable to L. We denote by $|\gamma_j| = O(L)$. Then

$$\frac{l_\rho(\gamma)}{l_{j_0}(\gamma)} \leq \frac{\sum_{i=1}^{N} |\gamma_i| - NC}{\sum_{i=1}^{N} |\gamma_i|} = 1 - \frac{C}{O(L)}$$

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Proposition

For any simple closed curve $\gamma \in \Pi$ that intersects λ on $\hat{S}$, the j_0-length of γ is strictly greater than its ρ-length, such that

$$\sup_{\gamma} \frac{l_\rho(\gamma)}{l_{j_0}(\gamma)} < 1$$

when γ varies over all simple closed curves on $S_{g,k}$ that intersect λ.

If λ is filling, then j_0 strictly dominates ρ.

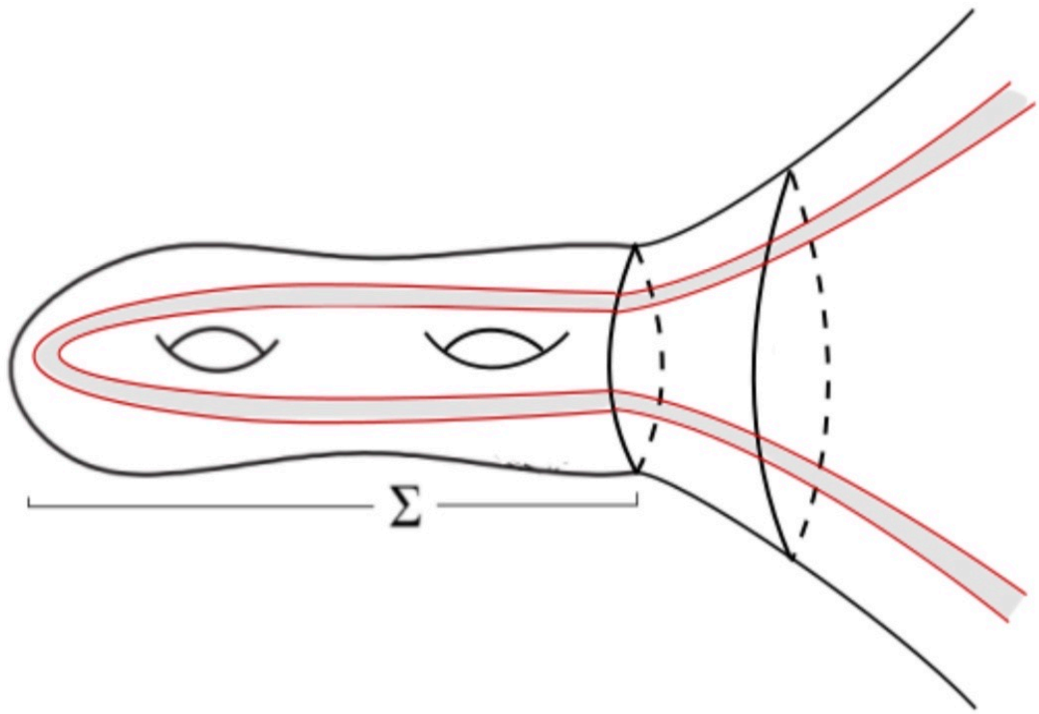
Non-filling case: modify j_0 to a strictly dominating representation j using *strip-deformations*.

Proposition

Given a hyperbolic surface Σ with non-empty geodesic boundary, there exists a hyperbolic surface Σ' homeomorphic to Σ, such that

$$\sup_{\gamma} \frac{l_{\Sigma}(\gamma)}{l_{\Sigma'}(\gamma)} < 1$$

where $l_X(\gamma)$ denotes the hyperbolic length of the (geodesic representative of) the curve γ on the hyperbolic surface X, and γ varies over all simple closed curves, including the boundary components.


Σ

Each connected component Σ^0 of $\widehat{S} \setminus \lambda$ (that is not simply-connected) is a crowned surface.

The convex core of Σ^0 is either a hyperbolic surface Σ with geodesic boundary, or a simple closed geodesic σ.

$$R = \widehat{S} \setminus (\Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_k \cup \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_l)$$