Negative curvature property of a Poisson–Kähler fibration

– Joint work with Xue-yuan Wan

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Negative curvature property

Problem (Negative curvature problem (NCP))

Let $p : \mathcal{X} \to \mathcal{B}$ be a proper holomorphic submersion between two Kähler manifolds. Assume that the Kodaira–Spencer map is injective. Does there exist a Kähler metric, say ω, on $\mathcal{B}$ satisfying the following negative curvature (NC) property ?

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Remark (Known cases)

Riemann surface case and Calabi–Yau case.

Known cases

(1) $\dim X_t = 1$, ω: Weil–Petersson metric. NC is known as the *Ahlfors theorem* proved by Ahlfors, Royden, Wolpert etc.

(2) $\dim X_t = \dim \mathcal{B} = 1$ and $\mathcal{B}$ is compact: *Kodaira–fibration* satisfying $g(X_t) \geq 3, \ g(\mathcal{B}) \geq 2$.

(3) K_{X_t}: Hermitian flat, NC follows from variation of Hodge theory (for trivial K_{X_t}) and Higgs bundle package (for general case).

(4) $K_{X_t} > 0$ and $\dim \mathcal{B} = 1$: To-Yeung, Schumacher, Berndtsson-Paun-Wang (iterated Kodaira-Spencer map).

(5) weak version of NCP (existence of Viehweg–Zuo sheaf and hyperbolicity): Viehweg-Zuo, Popa-Schnell, Deng etc.

Our main result is based on (3), (4), the background is Burns' local NC property along the leaves of a Monge–Ampère foliation.

Poisson–Kähler fibration

Definition (also called Monge–Ampère foliation (fibration))

A proper holomorphic submersion $p : (\mathcal{X}, \omega_{\mathcal{X}}) \to (\mathcal{B}, \omega_{\mathcal{B}})$ *between two Kähler manifolds is said to be Poisson–Kähler if*

$$(\omega_{\mathcal{X}} - p^* \omega_{\mathcal{B}})^{n+1} \equiv 0 \qquad (1)$$

on $\mathcal{X}$, *where* n *denotes the dimension of the fibers.*

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Theorem (Theorem A)

Poisson–Kähler fibration satisfies the nagative curvature property.

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Remark

Theorem A is also proved independently without using Higgs bundles by Berndtsson. We use the same Weil–Petersson type metric but different approaches for the curvature computation.

Examples

(1) Trivial fibrations: complexification of Kähler metric geodesics, holomorphic curves in the complex Banach manifold

$$\{f \in \mathrm{Diff}^k(X,\omega) : (f^{-1})^*\omega \text{ is Kähler}\}.$$

(2) Non-trivial fibrations: torus family with Kähler total space.

Family of elliptic curves: For each t in the upper half plane $\mathbb{H}$,

$$X_t := \mathbb{C}/(\mathbb{Z} + t\mathbb{Z}).$$

The $\mathbb{R}$-linear quasi-conformal mapping $f^t : \mathbb{C} \to \mathbb{C}$ defined by

$$f^t(1) = 1, \quad f^t(t) = i, \tag{2}$$

naturally induces a map, still denoted by f^t, from X_t to X_i. Put

$$f : \mathcal{X} \to \mathbb{H} \times X_i, \quad f(t,\zeta) := (t, f^t(\zeta)),$$

where $\mathcal{X} := \{X_t\}_{t\in\mathbb{H}} \simeq (\mathbb{H} \times \mathbb{C})/\mathbb{Z}^2$.

Family of elliptic curves

Put

$$\omega := f^*(idz \wedge d\bar{z}), \quad A := \frac{\zeta - \bar{\zeta}}{\bar{t} - t},$$

we compute

$$\omega = \frac{i}{\operatorname{Im} t} \left(d\zeta \wedge d\bar{\zeta} + A\, d\zeta \wedge d\bar{t} + A\, dt \wedge d\bar{\zeta} + A^2 dt \wedge d\bar{t} \right).$$

Thus ω is of degree-$(1,1)$ satisfying $\omega^2 = 0$ and

$$p : (\mathcal{X}, \omega) \to \mathbb{H}$$

is Poisson–Kähler, moreover the natural $SL_2(\mathbb{Z})$ action

$$SL_2(\mathbb{Z}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} : (t, \zeta) \mapsto \left(\frac{at+b}{ct+d}, \frac{\zeta}{ct+d} \right),$$

preserves ω.

Properties of Poisson–Kähler fibrations

Definition (Horizontal vector fields)

Let $p : (\mathcal{X}, \omega) \to \mathcal{B}$ be a relative Kähler fibration. Vector field V on $\mathcal{X}$ is horizontal if

$$\omega(V, W) = 0$$

for W tangent to the fibers.

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A horizontal V *is the horizontal lift of* $v \in T_{\mathcal{B}}$ *if* $p_* V = v$.

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Remark (Integrability of horizontal distribution)

The horizontal distribution is integrable iff the horizontal lift $\{V_j\}$ *of local basis* $\{\partial/\partial t^j\}$ *of* $T\mathcal{B}$ *satisfies*

$$[V_j, V_k] = [V_j, \overline{V_k}] = 0.$$

Properties of Poisson–Kähler fibrations

Proposition (Integrability VS Poisson–Kähler)

The horizontal distribution of a relative Kähler fibration $p : (\mathcal{X}, \omega) \to \mathcal{B}$ is integrable iff there is a real d-closed smooth $(1,1)$-form α on $\mathcal{B}$ such that

$$(\omega - p^*\alpha)^{n+1} \equiv 0.$$

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Proposition (Poisson-Kähler=Poisson+Kähler)

A Kähler fibration $p : (\mathcal{X}, \omega_{\mathcal{X}}) \to (\mathcal{B}, \omega_{\mathcal{B}})$ is a Poisson map iff it is Poisson-Kähler.

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Remark (Poisson map)

"Poisson map" means $\{p^*f, p^*g\}_{\omega_{\mathcal{X}}} = p^*\{f, g\}_{\omega_{\mathcal{B}}}$ for all smooth functions f, g on $\mathcal{B}$.

Non-harmonic Weil-Petersson metric

$p : (\mathcal{X}, \omega) \to \mathcal{B}$ relative Kähler fibration, V horizontal lift of $\partial/\partial t$.

We call $\kappa := (\overline{\partial} V)|_{X_t}$ the ω-Kodaira–Spencer tensors on X_t.

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Definition (Non-harmonic Weil–Petersson metric)

We call

$$|\partial/\partial t|_{DF}^2 := ||\kappa||^2 := \int_{X_t} |\kappa|_{\omega_t}^2 \frac{\omega_t^n}{n!}, \quad \omega_t := \omega|_{X_t},$$

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Remark

In the above definition, we do NOT use the harmonic part of κ. We use the notation "DF" since it reduces to the Donaldson-Fujiki metric in the Poisson–Kähler case.

Main idea: For smooth family of form u^t, denote by

$$||u||(t) := ||u^t||$$

the fiberwise L^2-norm. If $d\omega = 0$ then

$$||u||_t^2 = (D_t u, u) + (u, D_{\bar{t}} u), \quad D_t := [\partial, \delta_V], \quad D_{\bar{t}} := [\bar{\partial}, \delta_{\bar{V}}],$$

moreover, $D_{\bar{t}} u = 0$ ($\{u^t\}$ is a holomorphic section) implies

$$||u||_{t\bar{t}}^2 = ||D_t u||^2 - (\Theta_{t\bar{t}} u, u).$$

Recall that if D is the Chern connection of a bundle V then it induces a Chern connection $\mathbf{D}$ on $\mathrm{End}\, V$ by $\mathbf{D} f := [D, f]$. In our case, think of κ as an endomorphism

$$\kappa \cdot u := [\bar{\partial}, \delta_V] u, \text{ known as the Kodaira-Spencer action}$$

we know that $[D_{\bar{t}}, \kappa] = 0$ implies that

$$||\kappa||_{t\bar{t}}^2 = ||[D_t, \kappa]||^2 - ([\Theta_{t\bar{t}}, \kappa], \kappa).$$

$[D_{\bar{t}}, \kappa] = 0$??, since $D_{\bar{t}} = [\bar{\partial}, \delta_{\bar{V}}]$, $\kappa = [\bar{\partial}, \delta_V]$, we know that $[D_{\bar{t}}, \kappa]$ is the degree $(-1, 1)$ part of

$$[L_{\bar{V}}, L_V] = L_{[V, \bar{V}]},$$

it vanishes iff $[V, \bar{V}] = 0$, i.e. $\{V, \bar{V}\}$ is integrable (more or less equivalent to Poisson–Kähler). Moreover, denote by ∇ the connection defined by the full Lie derivatives, i.e.

$$\nabla = D + \theta + \bar{\theta}, \quad \theta := dt \otimes \kappa.$$

Variation of non-harmonic Weil–Petersson metrics

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Observation (Higgs flat=integrable=Poisson-Kähler)

∇ is Higgs flat iff the horizontal distribution is integrable iff p is Poisson-Kähler up to a factor from the base, in which case the Higgs package gives the negative curvature properties.

Higgs package

Higgs description explains

$$||\kappa||^2_{t\bar{t}} = ||[D_t, \kappa]||^2 - ([\Theta_{t\bar{t}}, \kappa], \kappa),$$

and gives $\Theta_{t\bar{t}} = -[\kappa, \bar{\kappa}]$, thus

$$||\kappa||^2_{t\bar{t}} \geq ([[\kappa, \bar{\kappa}], \kappa], \kappa) = ||[\kappa, \bar{\kappa}]||^2 \geq \frac{2||\kappa||^4}{n|X_t|},$$

gives negativity of holomorphic sectional curvature. Similarly,

$$\frac{\partial^2}{\partial t^j \partial \bar{t}^j} ||\kappa_I||^2 \geq \frac{2\,|(\kappa_I, \kappa_j)|^2}{n|X_t|},$$

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Remark (General relative Kähler fibration)

The computation also applies to general cases (with some extra curvature terms).

Relation to polystable vector bundles

Theorem

Let E be a holomorphic vector bundle over a compact Kähler manifold $\mathcal{B}$. Let $P(E) := (E \setminus \{0\})/\mathbb{C}^*$ be the projectivization of E. Then the followings are equivalent:

1) *(E, h) is projectively flat, i.e. there exists a hermitian metric h on E such that $\Theta(E, h) = \alpha \otimes Id_E$;*
2) *the natural projection $P(E) \to \mathcal{B}$ is Poisson–Kähler.*

In case $\dim \mathcal{B} = 1$, *both are equivalent to polystability of E.*

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Remark

Aikou proved that E admits a projectively flat Hermitian metric iff $P(E) \to \mathcal{B}$ is flat Kähler, i.e. locally biholomorphic to product Kähler fibrations. Our proof depends on Berndtsson curvature formula of the direct image sheaf.

Non-trivial Poissson–Kähler fibrations

Locally there are many non-trivial Poissson–Kähler fibrations.

Theorem (Arezzo–Tian's theorem)

Every relative Kähler fibration is locally Poisson–Kähler, i.e. every point in the base manifold has a neighborhood over which the fibration possesses a Poisson–Kähler structure.

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Remark (Motivation)

Moduli fibration of abelian varieties (which is Poisson-Kähler outside a proper subvariety) suggests to study the following:

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Remark (Final remark)

Thanks for your patience!