Plurisubharmonicity and convexity of energy functions on Teichmüller space

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Energy function

- Let $\pi : \mathcal{X} \to \mathcal{T}$ be Teichmüller curve over Teichmüller space $\mathcal{T}$ of a surface Σ, $g_\Sigma \geq 2$, namely it is the holomorphic family of Riemann surfaces over $\mathcal{T}$, the fiber $\mathcal{X}_z := \pi^{-1}(z)$ being exactly the Riemann surface given by the complex structure $z \in \mathcal{T}$.
- Let (M^n, g) be a closed Riemannian manifold and $u_0 : (M^n, g) \to (\mathcal{X}_z, \Phi_z)$ a continuous map, where Φ_z is the hyperbolic metric on the Riemann surface $\mathcal{X}_z$.
- For each $z \in \mathcal{T}$, there exists a smooth harmonic map $u : (M^n, g) \to (\mathcal{X}_z, \Phi_z)$ homotopic to u_0 (Eells-Sampson), and it is unique unless the image of the map is a point or a closed geodesic (Hartman).

The energy function is defined by

$$E(z) = E(u(z)) = \frac{1}{2}\int_M |du(z)|^2 d\mu_g, \quad z \in \mathcal{T}, \tag{1}$$

which is a smooth function on Teichmüller space $\mathcal{T}$.

- $M = S^1$, $\sqrt{E} = \ell(\gamma)$ is the geodesic length function of a closed curve γ. In 1987, Wolpert showed the function is strictly plurisubharmonic and strictly convex along WP geodesics.
- In 2012, Wolf presented a precise formula for the second derivative of $\ell(\gamma)$ along a Weil-Petersson geodesic.
- By using the methods of Kähler geometry, Axelsson and Schumacher (in 2012) obtained the formulas for the first and the second variation of $\ell(\gamma)$, and proved that its logarithm $\log \ell(\gamma)$ is strictly plurisubharmonic.

- If M is a Riemann surface and the fixed map u_0 is identity, in 1989, Wolf defined the energy function and proved the energy function is proper, strictly convex along WP geodesics.
- For a general Riemannian manifold M, in 1999, Yamada proved the strict convexity of the energy function along the WP geodesics.

Inspired by the above results. Naturally, one may ask whether the logarithm of energy function is also strictly plurisubharmonic for a general Riemannian manifold?

By following Axelsson-Schumacher's method, we obtain

Theorem (Kim-W.-Zhang, 2018)

Let $\pi : \mathcal{X} \to \mathcal{T}$ be Teichmüller curve over Teichmüller space $\mathcal{T}$. Let (M^n, g), be a Riemannian manifold and fix a continuous map $u_0 : (M^n, g) \to (\mathcal{X}_z, \Phi_z)$. Consider the energy $E(z)$ of the harmonic map from (M^n, g) to $\mathcal{X}_z = \pi^{-1}(z)$ homotopic to u_0, $z \in \mathcal{T}$. Then the logarithm of energy $\log E(z)$ is a strictly plurisubharmonic function on Teichmüller space. In particular, the energy function is also strictly plurisubharmonic.

Variations of energy function

Let $\pi : \mathcal{X} \to \mathcal{T}$ be a Teichmüller curve. Denote by (z^α, v) the holomorphic coordinates of $\mathcal{X}$ such that $z^\alpha, 1 \leq \alpha \leq 3g-3$ are the holomorphic coordinates of $\mathcal{T}$, and v is holomorphic coordinate of each fiber. Each fiber is equipped with the hyperbolic metric

$$\omega_{X_z} = \sqrt{-1}\phi_{v\bar{v}} dv \wedge d\bar{v}.$$

One can define the following tensor

$$A_\alpha = A_{\alpha\bar{v}\bar{v}} \overline{u_i^v} \phi^{v\bar{v}} dx^i \otimes \frac{\partial}{\partial v} \in A^1(M, u^* T\mathcal{X}_z); \tag{2}$$

where $A^v_{\alpha\bar{v}} = \partial_{\bar{v}} \left(-\phi^{v\bar{v}} \phi_{\alpha\bar{v}} \right), A_{\alpha\bar{v}\bar{v}} = A^v_{\alpha\bar{v}} \phi_{v\bar{v}}, \phi^{v\bar{v}} := (\phi_{v\bar{v}})^{-1}$. The first variation of the energy function $E(z)$ is given by

$$\frac{\partial}{\partial z^\alpha} E(z) = \langle A_\alpha, \partial u \rangle = \int_M A_{\alpha\bar{v}\bar{v}} \overline{u_i^v u_j^v} g^{ij} d\mu_g. \tag{3}$$

where $\partial u := u_l^v dx^l \otimes \frac{\partial}{\partial v}$.

Let Φ_z be the hyperbolic Riemannian metric on $\mathcal{X}_z$. Denote $T_{\mathbb{C}}\mathcal{X}_z = T\mathcal{X}_z \oplus \overline{T\mathcal{X}_z}$ the complex tangent bundle. Then there is a natural connection on $\wedge^\ell T^*M \otimes u^* T_{\mathbb{C}}\mathcal{X}_z$ induced from the Levi-Civita connections of (M^n, g) and $(\mathcal{X}_z, \Phi_z)$. Let $\Delta = \nabla\nabla^* + \nabla^*\nabla$ be the Hodge-Laplace operator on $A^\ell(M, u^*T\mathcal{X}_z)$, and set

$$\mathcal{L} = \Delta + \frac{1}{2}|du|^2, \quad \mathcal{G} = g^{ij}\phi_{v\bar{v}} u_i^v u_j^v \frac{\partial}{\partial v} \otimes d\bar{v} \in \mathrm{Hom}(u^*\overline{T\mathcal{X}_z}, u^*T\mathcal{X}_z), \tag{4}$$

and $c(\phi)_{\alpha\bar{\beta}} := \phi_{\alpha\bar{\beta}} - \phi_{\alpha\bar{v}}\phi_{v\bar{\beta}}\phi^{v\bar{v}}$ denotes the geodesic curvature. Then the second variation of the energy function is given by

$$\partial_\alpha\partial_{\bar{\beta}} E = \frac{1}{2}\int_M c(\phi)_{\alpha\bar{\beta}}|du|^2 d\mu_g + \langle (Id - \nabla\left(\mathcal{L} - \mathcal{G}\mathcal{L}^{-1}\overline{\mathcal{G}}\right)^{-1}\nabla^*)A_\alpha, A_\beta\rangle. \tag{5}$$

The case of dim $M = 1$

If dim $M = 1$, then

$$\partial_\alpha \partial_{\bar\beta} E^{1/2} = \frac{1}{2}\frac{1}{E^{1/2}}\Big(\int_M (\Box + 1)^{-1}(A_\alpha, A_\beta) d\mu_g + \langle \frac{1}{2}|du|^2(|du|^2 + \Delta)^{-1} A_\alpha, A_\beta\rangle\Big), \quad (6)$$

where $\Box = -\phi^{v\bar v}\partial_v \partial_{\bar v}$. In this case, $|du| =$ constant. And we used Schumacher's formula $(1 + \Box)c(\phi)_{\alpha\bar\beta} = A^v_{\alpha\bar v}\overline{A^v_{\beta\bar v}}$.

Corollary (Axelsson-Schumacher, 2012)

If we take the arc-length parametrization at $z = z_0$, i.e. $\frac{1}{2}|du|^2(z_0) = 1$, then

$$\partial_\alpha \partial_{\bar\beta} \ell(z)|_{z=z_0} = \frac{1}{2}\Big(\int_M (\Box + 1)^{-1}(A_\alpha, A_\beta) d\mu_g + \langle (2 + \Delta)^{-1} A_\alpha, A_\beta\rangle\Big). \quad (7)$$

Plurisubharmonicity

For higher dimensional M, $\nabla^2 \not\equiv 0$ generally, and we shall treat the second term in more details. A major ingredient of our proof is the following decomposition

$$Id - \nabla\left(\mathcal{L} - \mathcal{G}\mathcal{L}^{-1}\overline{\mathcal{G}}\right)^{-1}\nabla^* = (\Delta^{-1}\Delta - \nabla\Delta^{-1}\nabla^*) + (\nabla\Delta^{-1}\nabla^* - \nabla\left(\mathcal{L} - \mathcal{G}\mathcal{L}^{-1}\overline{\mathcal{G}}\right)^{-1}\nabla^*) + \mathbb{H},$$

where $\mathbb{H}$ is the orthogonal projection onto harmonic forms. And we can prove the first two operators are non-negative when acting on $A^1(M, u^*T\mathcal{X}_z)$. Thus

$$\partial_z\partial_{\bar{z}}E(z) \geq \frac{1}{2}\int_M c(\phi)_{z\bar{z}}|du|^2 d\mu_g + ||\mathbb{H}(A)||^2 \tag{8}$$

For simplify, we assume the base is dimension one.

Note that $\partial u := u_l^v dx^l \otimes \frac{\partial}{\partial v}$ is a harmonic element (i.e. $\nabla(\partial u) = \nabla^*(\partial u) = 0$) since u is a harmonic map. We obtain a lower bound for $||\mathbb{H}(A)||^2$

$$||\mathbb{H}(A)||^2 \geq \frac{1}{||\partial u||^2} |\langle A, \partial u \rangle|^2 = \frac{1}{E} \frac{\partial E}{\partial z} \frac{\partial E}{\partial \bar{z}}, \tag{9}$$

Therefore,

$$\partial_z \partial_{\bar{z}} \log E(z) \geq \frac{1}{||du||^2} \int_M c(\phi)_{z\bar{z}} |du|^2 d\mu_g > 0, \tag{10}$$

which proves the strict plurisubharmonicity of logarithm of energy function.

Related to Weil-Petersson metric

Definition

The Weil-Petersson metric ω_{WP} on Teichmüller space $\mathcal{T}$ is defined by

$$\omega_{WP} = \sqrt{-1} G_{\alpha\bar{\beta}} dz^{\alpha} \wedge d\bar{z}^{\beta}, \quad G_{\alpha\bar{\beta}}(z) = \int_{\mathcal{X}_z} A^{v}_{\alpha\bar{v}} \overline{A^{v}_{\beta\bar{v}}} \sqrt{-1} \phi_{v\bar{v}} dv \wedge d\bar{v}. \tag{11}$$

Because $A^{v}_{\alpha\bar{v}} d\bar{v} \otimes \frac{\partial}{\partial v}$ is a harmonic Betrami differential.

Theorem (Kim-W.-Zhang, 2019)

Let (M, ω_g) be a compact Kähler manifold and fix a smooth map $u_0 : M \to \Sigma$, let $E(z)$ denote the energy function of harmonic maps from (M, g) to $(\mathcal{X}_z, \Phi_z)$ in the class $[u_0]$, where g is the Riemannian metric associated to ω_g. If $u(z_0)$ is holomorphic or anti-holomorphic for some $z_0 \in \mathcal{T}$, then

$$\sqrt{-1}\partial\bar{\partial} \log E(z)|_{z=z_0} = \frac{\omega_{WP}}{2\pi(g(\Sigma) - 1)}. \tag{12}$$

As a corollary, we obtain

Corollary

If M is a Riemann surface, and $u(z_0)$ is holomorphic or anti-holomorphic, then

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=z_0} = |\deg u(z_0)| \cdot 2\omega_{WP}. \tag{13}$$

Here $\deg u(z_0)$ is the degree of $u(z_0)$.

In this case,

$$\frac{\partial E}{\partial z^\alpha}|_{z=z_0} = 2\int_M A_{\alpha\bar{v}\bar{v}} \overline{u_i^v u_j^v} g^{\bar{j}i} d\mu_g = 0$$

$$E(z_0) = \left| \int_M u^* \omega_{X_{z_0}} \right| = |2\pi \deg(u^* K_{X_{z_0}})| = 4\pi(g_\Sigma - 1)|\deg u(z_0)|.$$

Thus $\partial\bar{\partial}E = E\partial\bar{\partial}\log E$. In particular, if $u(z_0)$ is the identity map, then

$$\sqrt{-1}\partial\bar{\partial}E(z)|_{z=z_0} = 2\omega_{WP},$$

which was proved by M. Wolf in 1989.

Geodesic convexity

For the geodesic convexity of energy function, Yamada proved

Theorem (Yamada, 1999)

The energy function $E(z)$ is strictly convex along any Weil-Petersson geodesic in $\mathcal{T}$.

In 2012, Wolf gave a precise asymptotic formula on the hyperbolic metrics associated to WP geodesics. By using this formula, we obtain

Proposition (Kim-W.-Zhang, 2018)

The function $E(z)^c, c > 5/6$ (resp. $c = 5/6$) is strictly convex (resp. convex) along a Weil-Petersson geodesic.

Energy function

- (N, g): a Riemannian manifold;
- ∇^N denotes the Levi-Civita connection of (N, g);
- Riemann curvature endomorphism $R \in A^2(N, \mathrm{End}(TN))$,

$$R(X, Y)Z = \nabla^N_X \nabla^N_Y Z - \nabla^N_Y \nabla^N_X Z - \nabla^N_{[X,Y]} Z.$$

  Denote $R(X, Y, Z, W) := -\langle R(X, Y)Z, W \rangle$.
- Sectional curvature: For $X, Y \in TN$

$$K(X \wedge Y) = \frac{R(X, Y, X, Y)}{\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2}.$$

- Hermitian sectional curvature: For $X, Y \in TN \otimes \mathbb{C}$, by complex mult-linear extension

$$K_{\mathbb{C}}(X \wedge Y) := \frac{R(X, Y, \overline{X}, \overline{Y})}{\|X\|^2 \|Y\|^2 - |\langle X, \overline{Y} \rangle|^2}.$$

Fix a continuous map $u_0 : \Sigma \to N$. If there is a unique harmonic map $u(z) : (\mathcal{X}_z, \Phi_z) \to (N, g)$ homotopic to u_0, then one gets a smooth map $u(z, v) : \mathcal{X} \to N$ and the energy

$$E(z) = E(u(z)) = \frac{1}{2} \int_{\mathcal{X}_z} |du(z)|^2 d\mu_{\Phi_z} \tag{14}$$

is a smooth function on $\mathcal{T}$.

- If N is also a negatively curved Riemann surface Tromba showed that this energy function is strictly plurisubharmonic.
- When N has non-positive Hermitian sectional curvature, Toledo proved that the energy function is also plurisubharmonic.

A natural question is whether the logarithm of energy function is also plurisubharmonic. We obtain

Theorem (Kim-W.-Zhang, 2019)

Let (N, g) be a Riemannian manifold with non-positive Hermitian sectional curvature and fix a smooth map $u_0 : \Sigma \to N$. If there is a unique harmonic map $u(z) : \mathcal{X}_z \to N$ in the homotopy class $[u_0]$ for each $z \in \mathcal{T}$, then

- *the reciprocal energy function $E(z)^{-1}$ is plurisuperharmonic, that is, $\sqrt{-1}\partial\bar{\partial}E(z)^{-1} \leq 0$. In particular, $\sqrt{-1}\partial\bar{\partial}\log E \geq 0$.*
- *Moreover, if (N, g) has strictly negative Hermitian sectional curvature and $d(u(z_0))$ is never zero on $\mathcal{X}_{z_0}$ for some $z_0 \in \mathcal{T}$, then $\log E(z)$ is strictly plurisubharmonic at z_0.*

Horizontal-Vertical decomposition

- Denote by $(z; v)$ the local coordinates of $\mathcal{X}$ with $\pi(z, v) = z$, where $\pi : \mathcal{X} \to \mathcal{T}$.
- Let $\sqrt{-1}\phi_{v\bar{v}}(z, v)dv \wedge d\bar{v}$ be a smooth family of hyperbolic metrics for some weight ϕ with $e^{\phi} = \phi_{v\bar{v}}$. Set $\omega = \sqrt{-1}\partial\bar{\partial}\phi$.
- H-V decomposition: $T\mathcal{X} = \mathcal{H} \oplus \mathcal{V}$ and $T^*\mathcal{X} = \mathcal{H}^* \oplus \mathcal{V}^*$

$$\mathcal{H} = \text{Span}\left\{\frac{\delta}{\delta z^{\alpha}} = \frac{\partial}{\partial z^{\alpha}} + a_{\alpha}^{v}\frac{\partial}{\partial v}, a_{\alpha}^{v} := -\phi_{\alpha\bar{v}}\phi^{v\bar{v}}\right\}, \quad \mathcal{V} = \text{Span}\left\{\frac{\partial}{\partial v}\right\},$$

$$\mathcal{H}^* = \text{Span}\{dz^{\alpha}\}, \quad \mathcal{V}^* = \text{Span}\{\delta v = dv - a_{\alpha}^{v}dz^{\alpha}\}.$$

- $\omega = c(\phi) + \sqrt{-1}\phi_{v\bar{v}}\delta v \wedge \delta\bar{v}$.

Let $\{u(z)\}_{z\in\mathcal{T}}$ be a smooth family of harmonic maps. We shall treat it as a smooth map u,

$$u : \mathcal{X} \to N, \quad (z, v) \mapsto u(z, v) := (u(z))(v). \tag{15}$$

Then

$$du = \partial u + \bar{\partial} u \in A^1(\mathcal{X}, u^* TN), \tag{16}$$

where $\partial u := \partial u^i \otimes \frac{\partial}{\partial x^i} \in A^{1,0}(\mathcal{X}, u^* TN)$ and $\bar{\partial} u = \overline{\partial u}$. Let

$$\langle \partial u \wedge \bar{\partial} u \rangle := g_{ij}(u(z, v))\partial u^i \wedge \bar{\partial} u^j \in A^{1,1}(\mathcal{X}) \tag{17}$$

denote the two-form on $\mathcal{X}$ obtained by combining the wedge product in $\mathcal{X}$ with the Riemannian metric $\langle , \rangle$ on $u^* TN$. The corresponding energy function can be written as

$$E(z) = \sqrt{-1} \int_{\mathcal{X}/\mathcal{T}} \langle \partial u \wedge \bar{\partial} u \rangle. \tag{18}$$

The variations of energy function $E(z)$ are

$$\partial E(z) = \sqrt{-1}\int_{\mathcal{X}/\mathcal{T}} \partial\langle \partial u \wedge \bar{\partial} u\rangle = \sqrt{-1}\int_{\mathcal{X}/\mathcal{T}} \left[\partial\langle \partial u \wedge \bar{\partial} u\rangle\right]^{(\delta v \wedge \delta \bar{v})} \tag{19}$$

$$\partial\bar{\partial} E(z) = \sqrt{-1}\int_{\mathcal{X}/\mathcal{T}} \partial\bar{\partial}\langle \partial u \wedge \bar{\partial} u\rangle = \sqrt{-1}\int_{\mathcal{X}/\mathcal{T}} \left[\partial\bar{\partial}\langle \partial u \wedge \bar{\partial} u\rangle\right]^{(\delta v \wedge \delta \bar{v})}. \tag{20}$$

- $\left[\partial\langle \partial u \wedge \bar{\partial} u\rangle\right]^{(\delta v \wedge \delta \bar{v})} = \langle \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u\rangle \wedge dz^\alpha$.
- Denote $\partial^V = \delta v \otimes \frac{\partial}{\partial v}$, and ∇ the induced connection on $(\mathcal{V}^* \oplus \overline{\mathcal{V}^*}) \otimes u^* TN$ from $(\mathcal{V}^*, e^{-\phi})$ and (N, g),

$$\left[\partial\bar{\partial}\langle \partial u \wedge \bar{\partial} u\rangle\right]^{(\delta v \wedge \delta \bar{v})} = -2R\left(\frac{\partial u}{\partial v}, \frac{\delta u}{\delta z^\alpha}, \frac{\partial u}{\partial \bar{v}}, \frac{\delta u}{\delta \bar{z}^\beta}\right)\delta v \wedge \delta\bar{v} \wedge dz^\alpha \wedge d\bar{z}^\beta$$
$$+ 2\langle \nabla_{\frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u\rangle \wedge dz^\alpha \wedge d\bar{z}^\beta. \tag{21}$$

By using Cauchy-Schwarz inequality, one has

Lemma

For any $\xi = \xi^\alpha \frac{\partial}{\partial z^\alpha} \in T_z\mathcal{T}$ *it holds*

$$\left|\xi^\alpha \frac{\partial E(z)}{\partial z^\alpha}\right|^2 \leq E(z) \cdot \int_{\mathcal{X}/\mathcal{T}} \langle \nabla_{\bar{\xi}^\beta \frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle. \tag{22}$$

Thus,

$$\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta \geq 2 \int_{\mathcal{X}/\mathcal{T}} \langle \nabla_{\bar{\xi}^\beta \frac{\delta}{\delta \bar{z}^\beta}} \partial^V u \wedge \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u \rangle \tag{23}$$

$$\geq \frac{2}{E} \left|\xi^\alpha \frac{\partial E(z)}{\partial z^\alpha}\right|^2 = \frac{2}{E} \frac{\partial E(z)}{\partial z^\alpha} \frac{\partial E(z)}{\partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta. \tag{24}$$

Since

$$\frac{\partial^2 E(z)^{-1}}{\partial z^\alpha \partial \bar{z}^\beta} = -\frac{1}{E^2}\left(\frac{\partial^2 E(z)}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{2}{E}\frac{\partial E(z)}{\partial z^\alpha}\frac{\partial E(z)}{\partial \bar{z}^\beta}\right),$$

so $E(z)^{-1}$ is plurisuperharmonic.
On the other hand,

$$\sqrt{-1}\partial\bar{\partial}\log E = -E\sqrt{-1}\partial\bar{\partial}E^{-1} + E^{-2}\sqrt{-1}\partial E \wedge \bar{\partial}E,$$

$$\sqrt{-1}\partial\bar{\partial}E = E\sqrt{-1}\partial\bar{\partial}\log E + E^{-1}\sqrt{-1}\partial E \wedge \bar{\partial}E,$$

so both $\log E(z)$ and $E(z)$ are plurisubharmonic.

Strictly plurisubharmonic

If (N, g) has strictly negative Hermitian sectional curvature and

$$\frac{\partial^2 \log E(z)}{\partial z^\alpha \partial \bar{z}^\beta} \xi^\alpha \bar{\xi}^\beta = 0.$$

then

$$\frac{\partial u}{\partial v} \wedge \xi^\alpha \frac{\delta u}{\delta z^\alpha} = 0, \quad \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u = 0. \tag{25}$$

If moreover, $du(z_0)$ never zero on $\mathcal{X}_{z_0}$, then $\frac{\partial u}{\partial v}$ is also never zero. So there exists a vector filed $W = W^v \frac{\partial}{\partial v} \in A^0(\mathcal{X}_{z_0}, T\mathcal{X}_{z_0})$ such that

$$\xi^\alpha \frac{\delta u}{\delta z^\alpha} = du(W). \tag{26}$$

For the second equation, we have

$$0 = \nabla_{\xi^\alpha \frac{\delta}{\delta z^\alpha}} \bar{\partial}^V u = \left(\partial_{\bar{v}} W^v - \xi^\alpha A^v_{\alpha\bar{v}}\right) u^i_v \delta\bar{v} \otimes \frac{\partial}{\partial x^i}, \tag{27}$$

where $A^v_{\alpha\bar{v}} = \partial_{\bar{v}} a^v_\alpha$. So

$$\xi^\alpha A^v_{\alpha\bar{v}} d\bar{v} \otimes \frac{\partial}{\partial v} = \bar{\partial} W \in A^{0,1}(\mathcal{X}_{z_0}, T\mathcal{X}_{z_0}). \tag{28}$$

This implies that

$$\rho\left(\xi^\alpha \frac{\partial}{\partial z^\alpha}\right) = \left[\xi^\alpha A^v_{\alpha\bar{v}} d\bar{v} \otimes \frac{\partial}{\partial v}\right] = [\bar{\partial} W] = 0 \in H^1(\mathcal{X}_{z_0}, T\mathcal{X}_{z_0}). \tag{29}$$

Since the Kodaira-Spencer map $\rho : T_{z_0}\mathcal{T} \to H^1(\mathcal{X}_{z_0}, T\mathcal{X}_{z_0})$ is injective, so $\xi = 0$. Thus $\log E(z)$ is strictly plurisubharmonic at z_0.

The relation to Weil-Petersson metric

Now we assume that (N, h) is a Hermitian manifold and $g = \operatorname{Re} h$. Then we obtain

Theorem (Kim-W.-Zhang, 2019)

If $u(z_0)$ is holomorphic (resp. anti-holomorphic) and totally geodesic on $\mathcal{X}_{z_0}$, then

$$\sqrt{-1}\partial\bar{\partial} \log E(z)|_{z=z_0} = \frac{\omega_{WP}}{2\pi(g_\Sigma - 1)}. \tag{30}$$

Convexity at critical points

In this talk, we focus on the case when the target manifold $N = S$ is also a fixed Riemann surface with genus $g_S \geq 2$. Then

Theorem (Kim-W.-Zhang, 2019)

Let $u_0 : \Sigma \to S$ be a smooth map with non-zero degree, and let $E(t)$ be the associated energy function on the Teichmüller space $\mathcal{T}$ of Σ. If $t_0 \in \mathcal{T}$ is a critical point of $E(t)$, then the energy function is convex at this point. If moreover the associated harmonic map u_{t_0} satisfies that du_{t_0} is never zero, then the energy function is strictly convex at $t_0 \in \mathcal{T}$.

Remark

In particular, for the case $u_0 = Id$, then the energy function has a unique critical point, the second derivative of energy function at the critical point is exactly given by the Weil-Petersson metric (Tromba).

Weil-Petersson geodesic

Fix a point $t_0 \in \mathcal{T}$ and let $g_0 = \lambda_0^2 dz d\bar{z}$ be the corresponding hyperbolic metric on Σ_{t_0}. Let $\Gamma(t)$, $(\Gamma(0) = t_0)$, be the Weil-Petersson geodesic arc with initial tangent vector given by the harmonic Beltrami differential $\mu = \frac{\bar{q}}{\lambda_0^2}\frac{d\bar{z}}{dz}$, where qdz^2 is a holomorphic quadratic differential on Σ_{t_0}. Then the associated hyperbolic metrics on Σ_t has the following Taylor expansion near $t = 0$ (M. Wolf, 2012),

$$\begin{aligned} g(t) =& \lambda_0^2 dz d\bar{z} + t(qdz^2 + \overline{qdz^2}) \\ &+ \frac{t^2}{2}\left(\frac{2|q|^2}{\lambda_0^4} - 2(\Delta - 2)^{-1}\frac{2|q|^2}{\lambda_0^4}\right)\lambda_0^2 dz d\bar{z} + O(t^4). \end{aligned} \tag{31}$$

Here $\Delta = \frac{4}{\lambda_0^2}\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{1}{\lambda_0^2}(\partial_x^2 + \partial_y^2)$.

Variation of energy function

Hence, the energy function along the Weil-Petersson geodesic $\Gamma(t)$ is

$$E(t) = \frac{1}{2}\int_{\Sigma} |du|^2 d\mu_{g_t} = \frac{1}{2}\int_{\Sigma} \mathrm{Tr}_{g(t)}(u^*(\rho^2 dv d\bar{v}))d\mu_{g_t}. \qquad (32)$$

The first derivative is

$$\frac{dE(t)}{dt}|_{t=0} = \mathrm{Re}\int_{\Sigma}\left(-\frac{4\rho^2}{\lambda_0^2}\bar{q}\overline{u_{\bar{z}}^v}u_z^v\right)\frac{i}{2}dz\wedge d\bar{z} = -4\left\langle \rho^2 u_z^v \overline{u_{\bar{z}}^v}dz^2, \frac{\bar{q}}{\lambda_0^2}\frac{d\bar{z}}{dz}\right\rangle_{QB},$$

where $\langle\cdot,\cdot\rangle_{QB}$ is a pairing between holomorphic quadratic differentials and harmonic Beltrami differentials. Thus, $t_0 \in \mathcal{T}$ is a critical point of energy function if and only if the associated harmonic map u_{t_0} is $\pm$ holomorphic.

Then the second derivative of energy function is given by

Proposition (Kim-W.-Zhang, 2019)

The second derivative of energy function is given by

$$\frac{1}{2}\frac{d^2E(t)}{dt^2}|_{t=0} = \int_\Sigma |du|^2 \frac{|q|^2}{\lambda_0^4} d\mu_{g_0} - \left\langle \mathcal{J}\left(\frac{\partial u}{\partial t}\right), \frac{\partial u}{\partial t}\right\rangle, \tag{33}$$

where $\mathcal{J}(\frac{\partial u}{\partial t}) = -Re(\frac{2\bar{q}}{\lambda_0^4}\nabla_z^{u^*T_{\mathbb{C}}S\otimes K_\Sigma}\frac{\partial u}{\partial z})$ *and*

$$\mathcal{J} := -\frac{1}{\lambda_0^2}\nabla_z\nabla_{\bar{z}} - \frac{1}{\lambda_0^2}R(\bullet, \frac{\partial u}{\partial z})\frac{\partial u}{\partial \bar{z}} \tag{34}$$

J is real, self-adjoint, semi-positive. Here the connection term $\nabla_z^{u^*T_{\mathbb{C}}S\otimes K_\Sigma}\frac{\partial u}{\partial z}$ *is the coefficient*

$$\nabla_{\partial/\partial z}^{u^*T_{\mathbb{C}}S\otimes K_\Sigma}\partial u = \nabla_{\partial/\partial z}^{u^*T_{\mathbb{C}}S\otimes K_\Sigma}(\frac{\partial u}{\partial z}\otimes dz) =: (\nabla_z^{u^*T_{\mathbb{C}}S\otimes K_\Sigma}\frac{\partial u}{\partial z})\otimes dz.$$

Denote by $\nabla^{0,1}$ the $(0,1)$-part of the connection ∇. Denote

$$\Delta^{0,1} := \nabla^{0,1}(\nabla^{0,1})^* + (\nabla^{0,1})^*\nabla^{0,1}. \tag{35}$$

In terms of $\Delta^{0,1}$ the Jacobi operator $\mathcal{J}$ on smooth sections of $u^*T_{\mathbb{C}}S$ is

$$\mathcal{J} = \Delta^{0,1} + \mathcal{R}, \tag{36}$$

where $\mathcal{R}(\bullet) := -\frac{1}{\lambda_0^2}R(\bullet, \frac{\partial u}{\partial z})\frac{\partial u}{\partial \bar{z}}$, $\mathcal{R}$ is semi-positive. Denote by $\mu = \frac{\bar{q}}{\lambda_0^2}d\bar{z} \otimes \frac{\partial}{\partial z}$ the harmonic Beltrami differential. Then

$$(\nabla^{0,1})^* i_\mu du = -\frac{1}{\lambda_0^2}\nabla_z(\frac{\bar{q}}{\lambda_0^2}\frac{\partial u}{\partial z}) = -\frac{\bar{q}}{\lambda_0^4}\nabla_z^{u^*T_{\mathbb{C}}S\otimes K_\Sigma}\frac{\partial u}{\partial z}. \tag{37}$$

The second derivative of energy function at $t = 0$ satisfies

$$\frac{d^2E(t)}{dt^2}|_{t=0} \geq 4\left(||\mathbb{H}(i_\mu du)||^2 - \mathrm{Re}\langle(\nabla^{0,1})^* i_\mu du, \mathcal{J}^{-1}(\overline{(\nabla^{0,1})^* i_\mu du})\rangle\right), \quad (38)$$

with the equality if and only if $\mathcal{R}(\mathcal{J}^{-1}(\nabla^{0,1})^* i_\mu du) = 0$, where $\mathbb{H}$ denotes the harmonic projection to the space $\mathrm{Ker}\Delta^{0,1}$
If $t = 0$ is a critical point, then $(\nabla^{0,1})^* i_\mu du // \frac{\partial u}{\partial z}, \mathcal{J}^{-1}((\nabla^{0,1})^* i_\mu du) // \frac{\partial u}{\partial z}$ and so

$$\mathcal{R}(\mathcal{J}^{-1}(\nabla^{0,1})^* i_\mu du) = 0 = \langle(\nabla^{0,1})^* i_\mu du, \mathcal{J}^{-1}(\overline{(\nabla^{0,1})^* i_\mu du})\rangle$$

we have

$$\frac{d^2E(t)}{dt^2}|_{t=0} = 4||\mathbb{H}(i_\mu du)||^2 \geq 0, \quad (39)$$

i.e. the energy is convex at the critical point $t_0 \in \mathcal{T}$. If moreover, du_{t_0} is never zero, then we can prove $\mathbb{H}(i_\mu du) \neq 0$, and so $\frac{d^2E(t)}{dt^2}|_{t=0} = 4||\mathbb{H}(i_\mu du)||^2 > 0$.

As an application , we have

Corollary

If $u_0 : \Sigma \to S$ *is a covering map, then there exists a unique complex structure* $t_0 \in \mathcal{T}$ *such that the associated harmonic map* u_{t_0} *is* $\pm$ *holomorphic, and*

$$E(t) \geq E(t_0) = Area(\Sigma). \tag{40}$$

Moreover, the energy density satisfies $\frac{1}{2}|du|^2(t_0) \equiv 1$. *Indeed, the unique hyperbolic metric on* Σ *which minimizes the energy is the pull-back hyperbolic metric via* u_{t_0}. *In this case,*

$$\frac{d^2E(t)}{dt^2}|_{t=0} = 4\|\mu\|^2_{WP} > 0. \tag{41}$$

For a general smooth and surjective map $u_0 : \Sigma \to S$, whether the critical points of energy function are unique?

Definition (Simple branched covering)

A branched covering $u : \Sigma \to S$ of absolute degree (maximum cardinality of a fiber) $n \geqslant 2$ is simple provided that for each $y \in S$ the fiber $u^{-1}(y)$ over y consists of at least $n-1$ points (and hence contains at most one singular point of local degree 2). A branched covering is called non-simple if it is not simple.

Proposition (Kim-W., 2020)

If $u_0 : \Sigma \to S$ is a non-simple branched covering, then the associated energy function has at least two critical points.

Proof.

- Since u_0 is a branched covering, then $u_0 : (\Sigma, [u_0^* h]) \to (S, h)$ is $\pm$-holo., and $[u_0^* h] \in \mathcal{T}(\Sigma)$ is a critical point.
- By a result of Berstein and Edmonds, u_0 is homotopic to a simple branched covering $u : \Sigma \to S$. So $[u^* h] \in \mathcal{T}(\Sigma)$ is also a critical point.
- If $[u^* h] = [u_0^* h]$, then $\exists\, f : (\Sigma, [u_0^* h]) \to (\Sigma, [u^* h])$ is biholo. and $f \in [Id]$, so $u \circ f$ is $\pm$-holo. and in $[u_0]$. By uniqueness, $u_0 = u \circ f$.
- Since u_0 is a non-simple while u is simple, so $\exists\, p$ s.t. the absolute degree $|\deg(u_0, p)| > 2$. Then

$$2 < |\deg(u_0, p)| = |\deg(u, f(p)) \cdot \deg(f, p)| = |\deg(u, f(p))| \leq 2,$$

contradiction. Thus $[u^* h] \neq [u_0^* h]$.

□

Thanks for your attention !