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A new uniform lower bound on Weil-Petersson distance

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Teichmüller theory and related topics

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Let S_g *be a closed surface of genus* g $(g \geq 2)$.

$\mathcal{T}_g = \mathcal{T}(S_g)$ *is the Teichmüller space of* S_g.

$\mathcal{M}_g = \mathcal{M}(S_g)$ *is the moduli space of* S_g.

Injectivity radius

Let $p \in S_g$ be fixed and any $X \in \mathcal{T}_g$. The *injectivity radius* $\mathrm{Inj}_X(p)$ of X at p is *half* of the length of a shortest nontrivial closed geodesic loop based at p.

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Let

$$\sigma : [0, 2\,\mathrm{Inj}_X(p)] \to X$$

be such a shortest geodesic loop with $\sigma(0) = \sigma(2\,\mathrm{Inj}_X(p)) = p$ of arc-length parameter. Then

1. the restriction $\sigma : [0, \mathrm{Inj}_X(p)] \to X$ is a minimizing geodesic;
2. the restriction $\sigma : [\mathrm{Inj}_X(p), 2\,\mathrm{Inj}_X(p)] \to X$ is also a minimizing geodesic.

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The map $\mathrm{Inj}_{(\cdot)}(p) : \mathcal{T}_g \to \mathbb{R}^{>0}$ is continuous.

Systole function

For any $X \in \mathcal{T}_g$, we let $\ell_{sys}(X)$, called the *systole* of X, denote the length of shortest closed geodesics in X. The systole function

$$\ell_{sys}(\cdot) : \mathcal{T}_g \to \mathbb{R}^+$$

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1. $\ell_{sys}(X) = 2 \min_{p \in X} \operatorname{Inj}_X(p)$.
2. $\ell_{sys}(X) \leq 2 \operatorname{Inj}_X(p) \leq 2 \ln(4g - 2)$.

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2. $\ell_{sys}(X) \leq 2\,\mathrm{Inj}_X(p) \leq 2 \ln(4g-2)$.

Theorem (Buser-Sarnak 1994)

There exists a universal constant $U > 0$, independent of g, such that for all $g \geq 2$,

$$\sup_{X \in \mathcal{T}_g} \ell_{sys}(X) \geq U \ln g.$$

Geodesic length function

For any essential closed curve $\alpha \subset S_g$ and $X \in \mathcal{T}_g$, there exists a unique closed geodesic $[\alpha]$ in X representing α. The *geodesic length function* of α

$$\ell_\alpha(\cdot) : \mathcal{T}_g \to \mathbb{R}^{>0}$$

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The geodesic length function $\ell_\alpha(\cdot)$ is real-analytic on $\mathcal{T}_g$.

Gradient of geodesic length function

Let $X \in \mathcal{T}_g$ and $\alpha \subset X$ be a simple closed geodesic. One may lift α onto the imaginary axis in $\mathbb{H}$ and denote by $A : z \to e^{\ell_\alpha(X)} \cdot z$ its deck transformation on $\mathbb{H}$.

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(Gardiner 1975) The Weil-Petersson gradient $\nabla \ell_\alpha(X)$ of the geodesic length function $\ell_\alpha(\cdot)$ at X can be expressed as

$$\nabla \ell_\alpha(X)(z) = \frac{2}{\pi} \sum_{E \in \langle A \rangle \backslash \Gamma} \frac{\overline{E}'(z)^2}{\overline{E}(z)^2 \rho(z)} \frac{d\overline{z}}{dz} \in T_X \mathcal{T}_g$$

where $\langle A \rangle$ is the cyclic group generated by A, Γ is the Fuchsian group of X and $\rho(z)|dz|^2 = \frac{|dz|^2}{Im(z)^2}$ is the hyperbolic metric on $\mathbb{H}$.

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(Labourie-Wentworth 2018) Generalized formula at the Fuchsian locus of Hitchin representations.

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2. (Chu 1976, Wolpert 1975) The space $\mathcal{T}_g$ is *incomplete*.
3. (Wolpert 1987) The space $\mathcal{T}_g$ is *geodesically convex*.

A new uniform lower bound

Theorem (W. 2020)

Fix a point $p \in S_g$ $(g \geq 2)$. *Then for any* $X, Y \in \mathcal{T}_g$,

$$\left| \sqrt{\mathrm{Inj}_X(p)} - \sqrt{\mathrm{Inj}_Y(p)} \right| \leq 0.3884 \, \mathrm{dist}_{wp}(X, Y)$$

where dist_{wp} *is the Weil-Petersson distance.*

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Remark

(Rupflin-Topping 2018) With the notations above,

$$\left| \sqrt{\mathrm{Inj}_X(p)} - \sqrt{\mathrm{Inj}_Y(p)} \right| \leq c(g) \, \mathrm{dist}_{wp}(X, Y)$$

where $c(g) > 0$ is a constant depending on g.

Application

Corollary

For any $X, Y \in \mathcal{T}_g$ $(g \geq 2)$,

$$\left|\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)}\right| \leq 0.5492\,\mathrm{dist}_{wp}(X, Y).$$

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Proof.

Without loss of generality, one may assume that $\ell_{sys}(X) \geq \ell_{sys}(Y)$.

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Without loss of generality, one may assume that $\ell_{sys}(X) \geq \ell_{sys}(Y)$. Let $\alpha \subset Y$ be a shortest closed geodesic.

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Proof.

Without loss of generality, one may assume that $\ell_{sys}(X) \geq \ell_{sys}(Y)$. Let $\alpha \subset Y$ be a shortest closed geodesic. So for any $p \in \alpha$, we have $2 \operatorname{Inj}_Y(p) = \ell_{sys}(Y)$ and $2 \operatorname{Inj}_X(p) \geq \ell_{sys}(X)$.

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$$\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)} \leq \sqrt{2\,\mathrm{Inj}_X(p)} - \sqrt{2\,\mathrm{Inj}_Y(p)}$$
$$\leq \sqrt{2} \times 0.3884 \, \mathrm{dist}_{wp}(X, Y) = 0.5492 \, \mathrm{dist}_{wp}(X, Y).$$

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1. *(W. 2016, 2019)* $\left|\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)}\right| \leq K\, \mathrm{dist}_{wp}(X, Y)$ *where* $K > 0$ *is a uniform constant independent of* g.

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1. *(W. 2016, 2019)* $\left|\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)}\right| \leq K\,\mathrm{dist}_{wp}(X, Y)$ *where* $K > 0$ *is a uniform constant independent of* g.
2. *(Bridgeman-Bromberg 2020)* $\left|\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)}\right| \leq \frac{\mathrm{dist}_{wp}(X,Y)}{2}$.

We say

$$f_1(g) \prec f_2(g) \quad or \quad f_2(g) \succ f_1(g)$$

if there exists a universal constant $C > 0$, independent of g, such that

$$f_1(g) \leq C \cdot f_2(g).$$

And we say

$$f_1(g) \asymp f_2(g)$$

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Weil-Petersson Inradius

The *Weil-Petersson inradius* $\text{InRad}(\mathcal{M}_g)$ of $\mathcal{M}_g$ is

$$\text{InRad}(\mathcal{M}_g) := \sup_{X \in \mathcal{M}_g} \text{dist}_{wp}(X, \partial\overline{\mathcal{M}}_g)$$

where $\partial\overline{\mathcal{M}}_g$ is the boundary of $\overline{\mathcal{M}}_g$ consisting of nodal surfaces.

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Theorem (W. 2016)

For $g \geq 2$,

$$\text{InRad}(\mathcal{M}_g) \asymp \sqrt{\ln g}.$$

Outline of proof

Upper bound: it follows by the following two properties.

(1). For any $X_g \in \mathcal{M}_g$,

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$$\mathrm{dist}_{wp}(X_g, \mathcal{M}_\alpha) \leq \sqrt{2\pi\ell_\alpha(X_g)}$$

where $\mathcal{M}_\alpha$ is the stratum of $\overline{\mathcal{M}}_g$ whose pinching curve is α.

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where $\mathcal{M}_\alpha$ is the stratum of $\overline{\mathcal{M}}_g$ whose pinching curve is α. By choosing $\alpha \subset X_g$ to be a systolic curve,

$$\mathrm{InRad}(\mathcal{M}_g) \leq \sup_{X_g \in \mathcal{M}_g} \mathrm{dist}_{wp}(X_g, \mathcal{M}_\alpha) \prec \sqrt{\ln g}.$$

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Lower bound: let $\mathcal{X}_g \in \mathcal{M}_g$ be a Buser-Sarnak surface, i.e.,

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Recall that

$$|\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(Y)}| \prec \mathrm{dist}_{wp}(X, Y)$$

which implies that

$$\sqrt{\ln g} \prec \mathrm{dist}_{wp}(\mathcal{X}_g, \partial\mathcal{M}_g) \leq \mathrm{InRad}(\mathcal{M}_g).$$

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Set

$$sys(g) = \max_{X\in\mathcal{M}_g} \ell_{sys}(X).$$

It is known that

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Remark

This result above is firstly obtained by Bridgeman-Bromberg in 2020. The ideas of both proofs are similar, but the estimations are different.

Outline of proof

As introduced above, we know that

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Step-1: Show that the systole function

$$\ell_{sys}(\cdot) : \mathcal{T}_g \to \mathbb{R}^{>0}$$

is piecewise smooth along Weil-Petersson geodesics.

Step-2: A formula of Riera in 2005 says that

$$\langle \nabla \ell_\alpha, \nabla \ell_\alpha \rangle_{wp}(X) = \frac{2}{\pi}(\ell_\alpha(X) + \sum_{C \in \{\langle A \rangle \backslash \Gamma / \langle A \rangle - id\}} (u \ln \frac{u+1}{u-1} - 2))$$

where $u = \cosh(\text{dist}_{\mathbb{H}}(\tilde{\alpha}, C \circ \tilde{\alpha}))$ and $\tilde{\alpha}$ is an axis in $\mathbb{H}$ for α.

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If we choose $\alpha \subset X$ with $\ell_\alpha(X) = \ell_{sys}(X)$, by using *two-dimensional hyperbolic geometry* we show that

Proposition

Let $X \in \mathcal{M}_g$ with $\ell_{sys}(X) \geq 8$. Then for any curve $\alpha \subset X$ with $\ell_\alpha(X) = \ell_{sys}(X)$ there exists a uniform constant $C > 0$ independent of g such that

$$\frac{1}{\sqrt{2\pi}} \leq ||\nabla \ell_\alpha^{\frac{1}{2}}(X)||_{wp} \leq \frac{1}{\sqrt{2\pi}} \sqrt{\left(1 + Ce^{-\frac{\ell_{sys}(X)}{8}}\right)}.$$

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In particular, $||\nabla \ell_{sys}^{\frac{1}{2}}(X)||_{wp} \sim \frac{1}{\sqrt{2\pi}}$ as $\ell_{sys}(X) \to \infty$.

Step-3 (Endgame): Let $X \in \mathcal{M}_g$ with $\ell_{sys}(X) = sys(g)$ and $\gamma : [0, s) \to \mathcal{M}_g$ be the Weil-Petersson geodesic of arc-length parameter with $\gamma(0) = X$ and $\gamma(s) \in \partial\mathcal{M}_g$, where $s = \text{InRad}(\mathcal{M}_g)$.

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$$\inf_{0 \leq t \leq r_g} \ell_{sys}(\gamma(t)) = \ell_{sys}(\gamma(r_g)) = \sqrt{sys(g)}.$$

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Then,

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Then,

$$|\sqrt{sys(g)} - \sqrt{\sqrt{sys(g)}}| = |\sqrt{\ell_{sys}(X)} - \sqrt{\ell_{sys}(\gamma(r_g))}|$$
$$= |\int_0^{r_g} \langle \nabla \ell_{sys}^{\frac{1}{2}}(\gamma(t)), \gamma'(t) \rangle_{wp} dt|$$

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$$= |\int_0^{r_g} \langle \nabla \ell_{sys}^{\frac{1}{2}}(\gamma(t)), \gamma'(t) \rangle_{wp} dt| \leq \int_0^{r_g} ||\nabla \ell_{sys}^{\frac{1}{2}}(\gamma(t))||_{wp} dt$$

Step-3 (Endgame): Let $X \in \mathcal{M}_g$ with $\ell_{sys}(X) = sys(g)$ and $\gamma : [0, s) \to \mathcal{M}_g$ be the Weil-Petersson geodesic of arc-length parameter with $\gamma(0) = X$ and $\gamma(s) \in \partial\mathcal{M}_g$, where $s = \text{InRad}(\mathcal{M}_g)$. Let $r_g \in [0, s]$ such that

$$\inf_{0 \le t \le r_g} \ell_{sys}(\gamma(t)) = \ell_{sys}(\gamma(r_g)) = \sqrt{sys(g)}.$$

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Thus,

$$\liminf_{g \to \infty} \frac{\text{InRad}(\mathcal{M}_g)}{\sqrt{sys(g)}} \ge \sqrt{2\pi}.$$

Weil-Petersson diameter

The *Weil-Petersson diameter* $\text{diam}_{wp}(\mathcal{M}_g)$ of the moduli space $\mathcal{M}_g$ is

$$\text{diam}_{wp}(\mathcal{M}_g) = \sup_{X \neq Y \in \mathcal{M}_g} \text{dist}_{wp}(X, Y).$$

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Theorem (Cavendish-Parlier 2012)

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Open questions:

1. $\mathrm{diam}_{wp}(\mathcal{M}_g) \asymp \sqrt{g} \ln g$?
2. Does $\lim_{g \to \infty} \frac{sys(g)}{\ln g}$ exist?

Recall

Theorem (W. 2020)

Fix a point $p \in S_g$ $(g \geq 2)$. *Then for any* $X, Y \in \mathcal{T}_g$,

$$\left| \sqrt{\mathrm{Inj}_X(p)} - \sqrt{\mathrm{Inj}_Y(p)} \right| \leq 0.3884 \, \mathrm{dist}_{wp}(X, Y)$$

where dist_{wp} *is the Weil-Petersson distance.*

Outline of proof

Let $c : [0, T_0] \to \mathcal{T}_g$ be a Weil-Petersson geodesic of arc-length parameter where $T_0 > 0$ is a fixed constant, and let $\sigma : [0, 2\,\mathrm{Inj}_{c(t_0)}(p)] \to c(t_0)$ be a shortest geodesic loop based at p.

Now we outline the proof as the following several steps.

Step-1: (Rupflin-Topping 2018) Show that the function $\mathrm{Inj}_{(\bullet)}(p) : \mathcal{T}_g \to \mathbb{R}^{>0}$ is *locally Lipschitz* along Weil-Petersson geodesics.

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Now we outline the proof as the following several steps.

Step-1: (Rupflin-Topping 2018) Show that the function $\mathrm{Inj}_{(\bullet)}(p) : \mathcal{T}_g \to \mathbb{R}^{>0}$ is *locally Lipschitz* along Weil-Petersson geodesics. If it is differentiable at $t = t_0 \in (0, T_0)$, then

$$\left| \frac{d}{dt} \mathrm{Inj}_{c(t)}(p) \Big|_{t=t_0} \right| \leq \frac{1}{4} \int_0^{2\,\mathrm{Inj}_{c(t_0)}(p)} \left| c'(t_0)(\sigma(s)) \right| ds.$$

We divide Step-2 into two cases.

Step-2-0: If $\mathrm{Inj}_{c(t_0)}(p) \leq arcsinh(1)$, it is *not* hard to see that for any $s \in [0, 2\,\mathrm{Inj}_{c(t_0)}(p)]$,

$$\left(\sqrt{2}-1\right)\mathrm{Inj}_{c(t_0)}(p) \leq \mathrm{Inj}_{c(t_0)}(\sigma(s)) \leq \mathrm{Inj}_{c(t_0)}(p) \leq arcsinh(1).$$

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Proposition (Bridgeman-W. 2019)

Let X be a closed hyperbolic surface and μ be a harmonic Beltrami differential on X. Then for any $p \in X$ with $\mathrm{Inj}_X(p) \leq arcsinh(1)$,

$$|\mu(p)|^2 \leq \frac{\int_X |\mu(z)|^2 \,\mathrm{dArea}(z)}{\mathrm{Inj}_X(p)}.$$

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By applying the Proposition above and Step-1 we get

$$\left|\frac{d}{dt}\sqrt{\mathrm{Inj}_{c(t)}(p)}\Big|_{t=t_0}\right| \leq 0.3884.$$

Step-2-1: If $\mathrm{Inj}_{c(t_0)}(p) > \mathit{arcsinh}(1)$, it is *not* hard to see that

$$\min_{s \in [0, 2\,\mathrm{Inj}_{c(t_0)}(p)]} \mathrm{Inj}_{c(t_0)}(\sigma(s)) \geq 0.2407.$$

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Proposition (Teo 2009)

Let X be a closed hyperbolic surface and μ be a harmonic Beltrami differential on X. Then for any $p \in X$,

$$|\mu(p)|^2 \leq C(r) \int_{B(p;r)} |\mu(z)|^2 \,\text{dArea}(z), \ \forall\ 0 < r \leq \text{Inj}_X(p)$$

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where the constant $C(\bullet)$ is a function of $\bullet$.

By applying the Proposition above, Step-1 and a Necklace type inequality we get

$$\left| \frac{d}{dt} \sqrt{\text{Inj}_{c(t)}(p)} \Big|_{t=t_0} \right| \leq 0.3454.$$

Proposition (Necklace type inequality)

Let X be a hyperbolic surface. For any $p \in X$ we let $\sigma : [0, 2\,\mathrm{Inj}_X(p)] \to X$ be a shortest nontrivial geodesic loop based at p. Assume that

$$\inf_{s \in [0, 2\,\mathrm{Inj}_X(p)]} \mathrm{Inj}_X(\sigma(s)) \geq 2\varepsilon_0$$

for some uniform constant $\varepsilon_0 > 0$. Then for any function $f \geq 0$ on X, we have

$$\int_0^{2\,\mathrm{Inj}_X(p)} \left(\int_{B(\sigma(s);\varepsilon_0)} f \,\mathrm{dArea} \right) ds \leq 12\varepsilon_0 \int_{\mathcal{N}_{\varepsilon_0}(\sigma)} f \,\mathrm{dArea}\,.$$

Where $B(\sigma(s);\varepsilon_0) = \{q \in X;\ \mathrm{dist}(q, \sigma(s)) < \varepsilon_0\}$ and $\mathcal{N}_{\varepsilon_0}(\sigma)$ is the ε_0-neighbourhood of σ, i.e.,

$$\mathcal{N}_{\varepsilon_0}(\sigma) = \{z \in X;\ \mathrm{dist}\,(x, \sigma([0, 2\,\mathrm{Inj}_X(p)])) < \varepsilon_0\}$$

Step-3 (Endgame): We apply the Fundamental Theorem of Calculus to get

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$$\begin{aligned} |\sqrt{\mathrm{Inj}_X(p)} - \sqrt{\mathrm{Inj}_Y(p)}| &= \left| \int_0^{\mathrm{dist}_{wp}(X,Y)} \frac{d}{dt} \left(\sqrt{\mathrm{Inj}_{c(t)}(p)} \right) dt \right| \\ &\leq \int_0^{\mathrm{dist}_{wp}(X,Y)} \left| \frac{d}{dt} \left(\sqrt{\mathrm{Inj}_{c(t)}(p)} \right) \right| dt \end{aligned}$$

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&\leq \int_0^{\mathrm{dist}_{wp}(X,Y)} \left| \frac{d}{dt}\left(\sqrt{\mathrm{Inj}_{c(t)}(p)}\right) \right| dt \\
&\leq 0.3884\, \mathrm{dist}_{wp}(X, Y)
\end{aligned}$$

as desired.

Thank you!