Factoring quasiconformal & quasisymmetric mappings

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This talk is a joint work with Prof. Zhengxu He and Prof. Xiaojun Huang.

Let $f : U \to V$ be a sense-preserving homeomorphism of domains $U,\ V \subseteq \mathbb{R}^n$. We define, for any $\zeta \in U$,

$$H_f(\zeta) = \limsup_{r \to 0^+} \frac{\max_{|z-\zeta|=r} |f(z) - f(\zeta)|}{\min_{|z-\zeta|=r} |f(z) - f(\zeta)|}.$$

Denote by

$$H(f) = \begin{cases} \infty, & \text{if } \sup_{\zeta \in U} H_f(\zeta) = \infty, \\ \\ ess \sup_{\zeta \in U} H_f(\zeta), & \text{if } \sup_{\zeta \in U} H_f(\zeta) \neq \infty, \end{cases}$$

the linear dilatation of f.
When $H(f) < \infty$, we call it quasiconformal.

Theorem

A quasiconformal homeomorphism $f : U \to V$ possesses the following properties:

- *f is A. C. L (Absolutely Continuous on Lines). Also it is differentiable with Jacobian $J_f(\zeta) > 0$ almost everywhere;*
- *$f^{-1} : V \to U$ is quasiconformal;*
- *For measurable set $E \subset U$, the measure $m(E) = 0$ implies that $m(f(E)) = 0$.*

Lemma

If A is an $(n \times n)$-real matrix with determinant $\det(A) > 0$, then there exist $P,\ Q \in SO(n)$ such that

$$P \cdot A \cdot Q = diag\,(\lambda_1, \lambda_2, \cdots, \lambda_n),$$

with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

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with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$.

That is, for any orientation preserving linear mapping $\mathbb{A} : \mathbb{R}^n \to \mathbb{R}^n$, there are orthogonal bases $\{v_1, v_2, \cdots, v_n\}$ and $\{w_1, w_2, \cdots, w_n\}$ such that the matrix A with respect to these bases is diagonal.

Proof:

Since the determinant $\det(A) > 0$, the symmetric matrix AA^T is positive definite. There exists $P \in SO(n)$ such that

$$P \cdot AA^T \cdot P^T = \text{diag}\,(\lambda_1^2, \lambda_2^2, \cdots, \lambda_n^2),$$

where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$. Denote

$$Q = A^T \cdot P^T \cdot \text{diag}\,(\lambda_1^{-1}, \lambda_2^{-1}, \cdots, \lambda_n^{-1}).$$

Then $Q^T \cdot Q = I_n$. Consequently,

$$P,\ Q \in SO(n) \ \text{ and } \ P \cdot A \cdot Q = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

Suppose that a quasiconformal map $f : U \to V$ is differentiable at point ζ with Jacobian $J_f(\zeta) > 0$.

Let $A = A(\zeta)$ be the Jacobian matrix of f.

Suppose that

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$$

are the eigenvalues of the matrix A.

(*i*) the *linear dilatation*. $H(A) = \dfrac{\max\{|A\zeta| : |\zeta| = 1\}}{\min\{|A\zeta| : |\zeta| = 1\}} = \dfrac{\lambda_1}{\lambda_n}$;

(*ii*) the *outer dilatation*. $K_O(A) = \dfrac{|A|^n}{|det(A)|} = \dfrac{\lambda_1^n}{\lambda_1\lambda_2\cdots\lambda_n}$;

(*iii*) the *inner dilatation*. $K_I(A) = \dfrac{|A^{\#}|^n}{|det(A)|^{n-1}} = \dfrac{\lambda_1\lambda_2\cdots\lambda_n}{\lambda_n^n}$,

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In geometric terms, $H(A)$ measures the eccentricity of the ellipsoid $A(S^{n-1})$, while $H_I(A)$ and $H_O(A)$ relate the volume of $A(B^n)$ to the volumes of the balls centered at the origin that are, respectively, inscribed in and circumscribed about $A(S^{n-1})$.

Obviously, using definitions we obtain the following symmetry relations:

$$H(A) = H(A^{-1}), \ \ K_O(A) = K_I(A^{-1}).$$

Furthermore, these dilatation functions are submultiplicative. That is, for any non-degenerate $(n \times n)$-matrices A and B,

$$H(AB) \leq H(A)H(B),$$

$$K_I(AB) \leq K_I(A)K_I(B),$$

$$K_O(AB) \leq K_O(A)K_O(B).$$

For the proofs we will use exterior algebra.

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Definition

$$H(f) = ess \sup_{\zeta \in D} H(Df(\zeta)),$$

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If one of the above conditions is not satisfied, then

$$H(f) = K_I(f) = K_O(f) = \infty.$$

From the definition, it follows that

$$1 \leq H(f) \leq K_I(f), \qquad 1 \leq H(f) \leq K_O(f), \tag{1}$$

$$1 \leq K_O(f) \leq H(f)^{n-1}, \qquad 1 \leq K_I(f) \leq H(f)^{n-1}. \tag{2}$$

In particular, when $n = 2$ these dilatations

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Theorem (analytic definition)

A homeomorphism f is quasiconformal if and only if one of the dilatations $H(f), K_I(f), K_O(f)$ is finite.

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3. Väisälä shows that if the given mapping f is an ACL^n-homeomorphism, then it is necessarily differentiable almost everywhere. in D.

Alternative definition

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For any ring domain $A \subset \mathbb{R}^n$, Γ_A denote the family of local rectifiable curves joining ∂A. The muduli of A

$$M(\Gamma_A) = \inf \int_A \rho^n dm,$$

where the infimum is taken over all non-negative Borel functions $\rho : A \to [0, \infty]$ with $\int_\gamma \rho ds \geq 1, \quad \forall \gamma \in \Gamma_A$.

For any homeomorphism $f : U \to V$, Väisälä gives

$$K_I(f) = \sup \frac{M(\Gamma_{f(A)})}{M(\Gamma_A)}, \quad K_O(f) = \sup \frac{M(\Gamma_A)}{M(\Gamma_{f(A)})},$$

where the suprema are taken over all ring domains $A \subset U$ with $\overline{A} \subset U$.

A dilatation is lower semicontinuous if $\{f : K(f) \leq K\}$ are closed under local uniform convergence. That is, if $f_k : U \to V_k,\ \ k = 1, 2, \cdots$ is a sequence of quasiconformal mappings which converges locally uniformly to a homeomorphism $f : U \to \mathbb{V}$, then $K(f) \leq \liminf_{k\to\infty} K(f_k)$.

Theorem

If $f_k : U \to V_k$ is a sequence of qc mappings locally uniformly converging to $f : U \to V$,

$$K_I(f) \leq \liminf_{i\to\infty} K_I(f_i), \quad K_O(f) \leq \liminf_{i\to\infty} K_O(f_i).$$

That is, f is either a constant or K-quasiconformal.

In contrast, the linear dilatation H is not lower semicontinous.

Theorem (T. Iwaniec)

For each dimension $n \geq 3$ and $K > 1$ there exists a sequence of quasiconformal mappings $f_k : \mathbb{R}^n \to \mathbb{R}^n$ converging uniformly to a linear quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$H(x, f_k) = K < H(x, f), \qquad a.e.\, \mathbb{R}^n, \quad k = 1, 2 \cdots,$$

It implies that the standard Teichmüller metric approach to topology on the spaces of deformations for hyperbolic manifolds of dimension 2 has no counterpart in dimensions greater than 2.

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For $n(\geq 3)$-dimensional 1-quasiconformal homeomorphisms, we have the following Liouville Theorem due to F. Gehring and Yu. Reshetnyak. Note that this result involves no priori differentiability hypotheses.

Theorem (Liouville Theorem)

An $n(\geq 3)$-dimensional quasiconformal homeomorphism $f : U \to V$ is 1-quasiconformal if and only if f is the restriction to U of a Möbius transformation, i.e. the composition of even reflections in $(n-1)$-spheres or planes.

Bi-lipschitz mapping

Bi-lipschitz mapping

Definition

A map is L-bi-Lipschitz if

$$L^{-1}\,|z-z'| \le |f(z)-f(z')| \le L\,|z-z'|, \quad \forall\, z,\ z' \in \mathbb{R}^n.$$

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The least L is the isometric dilatation.

Obviously, by definition, an L-bi-Lip mapping is L^2-quasiconformal.

The converse is not true. For example,

$$f(x) = |x|^{b-1}, \qquad b = K^{1-n},$$

is K-quasiconformal but not bi-Lipschitz in $\mathbb{R}^n$.

Minimal factorization

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For quasiconformal maps f_1, f_2 and $f = f_2 \circ f_1$, we always have

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When $K[f] = K[f_2] \cdot K[f_1]$, we call $f = f_2 \circ f_1$ is a minimal factorization in the dilatation f.

Suppose that f is a plane quasiconformal map with maximal dilatation K and $1 < K_1 < K$.

The Measurable Riemann Mapping Theorem tells us that a minimal factorization $f = f_2 \circ f_1$ always exists with

$$K(f_1) = K_1, \qquad K(f_2) = K/K_1.$$

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In fact, supposing f has Beltrami differential $\mu(z)$, we choose $f_1 : \Delta \to \Delta$ to be the quasi-conformal mapping with Beltrami differential $t \cdot \mu(z)$, where

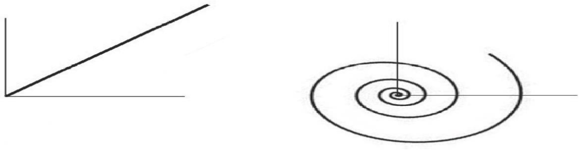
$$K_1 = \frac{1 + t|\mu|_\infty}{1 - t|\mu|_\infty}, \qquad K = \frac{1 + |\mu|_\infty}{1 - |\mu|_\infty}.$$

Then $f = (f \circ f_1^{-1}) \circ f_1$ is a minimal factorization.

Let $\lambda > 0$ and let

$$s_\lambda(\rho, \theta) = (\rho,\ \theta + \lambda \log \rho) : \mathbb{R}^2 \to \mathbb{R}^2$$

be the logarithmic spiral mapping, where $(\rho,\ \theta)$ are the polar coordinates of $\mathbb{R}^2$.



s_λ has Betrami differential

$$\mu_\lambda(z) = \frac{i\lambda}{2 + i\lambda} \cdot \frac{z}{\bar{z}},$$

with maximal dilatation

$$K = \frac{\sqrt{4 + \lambda^2} + \lambda}{\sqrt{4 + \lambda^2} - \lambda}.$$

Furthermore, the logarithmic spiral mapping s_λ is $\sqrt{K}$-bi-lipschitz.

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Using minimal factorization of quasiconformal map, the number of quasiconformal factors of s_λ with maximal dilatations $\leq L$ grows like

$$\log_L K.$$

On the other hand, we have

Theorem (M. Freedman & Z. He)

It requires at least $\frac{K}{\sqrt{L^2-1}}$ factors to write s_λ into a composition of L-bi-lipschitz homeomorphisms. That is, if

$$s_\lambda = f_m \circ \cdots \circ f_1,$$

where f_i is L-bi-lip, then the number m

$$\geq \frac{K}{\sqrt{L^2-1}}.$$

Thus for large K, the number of factors with small isometric dilatation needed to “unwind” the spiral map s_λ is much greater that the number of factors with the same linear dilatation.

■ M. Freedman & Z. He, Factoring the logarithmic spiral. Invent. Math. 92 no. 1 (1988) 129–138.

Define

$$f_{n,\lambda}(z,\ t_1,\cdots,\ t_{n-2}) = \left(s_{\lambda}(z),\ \frac{t_1}{\sqrt{K}},\cdots,\ \frac{t_{n-2}}{\sqrt{K}}\right).$$

where $z \in \mathbb{R}^2$ and $(t_1,\cdots,\ t_{n-2}) \in \mathbb{R}^{n-2}$.

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where $z \in \mathbb{R}^2$ and $(t_1,\cdots,\ t_{n-2}) \in \mathbb{R}^{n-2}$.

$f_{n,\lambda}$ is a quasiconformal homeomorphism with maximal distortion

$$K = H(f_{n,\lambda}) = K_I(f_{n,\lambda}),\ \ K^2 = K_O(f_{n,\lambda}).$$

Theorem

The n-dimensional quasiconformal map $f_{n,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ *admits no minimal factorizations in the linear dilatation. That is,*

$$f_{n,\lambda} \neq f_2 \circ f_1$$

for any quasiconformal map f_1 *with* $H(f_1) = K^s$ *and quasiconformal map* f_2 *with* $H(f_2) = K^{1-s}$*, where* $0 < s < 1$.

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Theorem

The n-dimensional quasiconformal map $f_{n,\lambda}$ *admits no minimal factorizations in the inner dilatation.*

By setting $g_{n,\lambda} \equiv f_{n,\lambda}^{-1} : \mathbb{R}^n \to \mathbb{R}^n$, we have

Theorem

The quasiconformal mapping $g_{n,\lambda}$ admits no minimal factorizations in the outer dilatation.

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Theorem

The quasiconformal mapping $g_{n,\lambda}$ admits no minimal factorizations in the outer dilatation.

Open Problem. For any $A > 1$, is there an n-dimensional ($n \geq 3$) quasiconformal map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $K(f) = K/A$ such that $f \neq f_2 \circ f_1$ for any quasiconformal map f_1 with $K(f_1) = K^s$ and quasiconformal map f_2 with $K(f_2) = K^{1-s}$?

■ Zhengxu He & L, Factorization of Higher Dimensional Quasiconformal Maps, Trans. A.M.S. 372 (2019), no. 8, 5341 – 5353.

Quasisymmetric maps

Quasisymmetric maps

Let (X, d), (Y, d') be metric spaces.

Definition (weak quasisymmetry)

Given a homeomorphism $f : X \to Y$, f is *quasisymmetric* if there is a constant $H < \infty$, for all $x \in X$ and all $r > 0$,

$$H_f(x, r) = \frac{\sup_{|y-x|=r} \left\{|f(y) - f(x)|\right\}}{\inf_{|y-x|=r} \left\{|f(y) - f(x)|\right\}} \leq H. \tag{3}$$

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Any quasiconformal mappings between $\mathbb{H}^2$ can be extended to a quasisymmetric self-mapping of the real line $\partial\mathbb{H}^2 = \mathbb{R}$.

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Any quasiconformal mappings between $\mathbb{H}^2$ can be extended to a quasisymmetric self-mapping of the real line $\partial\mathbb{H}^2 = \mathbb{R}$.

Conversely, quasisymmetric self-mapping of $\mathbb{R}$ can be extended to a quasiconformal self-mapping of $\mathbb{H}^2$.

Stronger quasisymmetry condition

Definition

A homeomorphism $f : X \to Y$ between two metric spaces is called η-*quasisymmetric* ($\eta - QS$) if there is a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{|f(x) - f(a)|}{|f(x) - f(b)|} \leq \eta\left(\frac{|x - a|}{|x - b|}\right) \tag{4}$$

for each triple x, a, b of points in X.

Obviously, (4) implies quasisymmetry as defined in (3). In general, these two notions are not equivalent.

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A metric space is called doubling if and only for all $0 < r < R$, there is $N = N(R/r)$ such that every open ball of radius R can be covered by N open balls of radius r.

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A metric space is called doubling if and only for all $0 < r < R$, there is $N = N(R/r)$ such that every open ball of radius R can be covered by N open balls of radius r.

Eg: $\mathbb{R}^n$, $n \geq 2$, these two definitions are equivalent.

Definition

For metric spaces, a homeomorphism $f : X \to Y$ is said *quasiconformal* if there is a constant $H < \infty$ such that

$$H_f(x) = \limsup_{r \to 0} H_f(x, r) \leq H \tag{5}$$

for all $x \in X$, where $H_f(x, r)$ is defined in (3).

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If f is η-quasisymmetric, then $H_f(x, r) = \frac{L_f(x,r)}{l_f(x,r)} \le \eta(1)$. So, quasisymmetric mappings are quasiconformal.

Conversely, If $f : X \to Y$ quasiconformal, X, Y Q-regular, X Lowner, Y-linearly locally connected, then f locally η-quasisymmetric.

Eg. If $f : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 2$, is quasiconformal, then it is quasisymmetric.

By a *curve* we mean any continuous mapping $\gamma : [a, b] \to X$. The *length* of γ is defined by

$$l(\gamma) = \sup \left\{ \sum_{i=1}^{n} |\gamma(t_i) - \gamma(t_{i+1})| \right\},$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \cdots < t_n = b$.

The curve is *rectifiable* if $l(\gamma) < \infty$.

Definition

For $c \geq 1$, a metric space X is *c-quasiconvex* if each pair of points $x, y \in X$ can be joined by an curve γ with length $l(\gamma) \leq c|x - y|$.

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Example: $\mathbb{S}^1$

Definition

For any $x \in G$, we denote by $\delta_G(x)$ the distance between x and the boundary of G. That is,

$$\delta_G(x) = \mathrm{dist}(x, \partial G).$$

Definition

Let γ be a rectifiable curve in an open set $G \subsetneq X$. The *quasihyperbolic length* of γ in G is

$$l_{qh}(\gamma) = \int_\gamma \frac{ds}{\delta_G(x)}.$$

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The quasihyperbolic distance between x and y in G is defined by

$$k_G(x,y) = \inf_{\gamma} l_{qh}(\gamma),$$

where γ runs over all rectifiable curves in G joining x and y. If there is no rectifiable curve in G joining x and y, we define

$$k_G(x,y) = +\infty.$$

If $G \subsetneq X$ is a rectifiably connected open set, it is clear that $k_G(x, y) < \infty$ for any two points $x, y \in G$. Thus it is easy to verify that $k_G(\cdot)$ is a metric in G, called the quasihyperbolic metric of G.

- **Upper half plane**

$\mathbb{H}^2 = \{z \in \mathbb{C} : \Im z > 0\}$. hyperbolic metric $d_H = \frac{ds}{\Im z}$.

$$\delta_{\mathbb{H}^2}(z) = \mathrm{dist}(z, \partial\mathbb{H}^2) = \Im z.$$

- **Unit disk**

$\Delta = \{z \in \mathbb{C} : |z| < 1\}$. hyperbolic metric $d_\Delta = \frac{2ds}{1-|z|^2}$.

$$1/2 \leq \frac{\delta_\Delta(z)}{1 - |z|^2} \leq 1.$$

By using the Schwarz Lemma, we have the following Schwarz-Picard Lemma.

Theorem

If $f : \mathbb{H}^2 \to \mathbb{H}^2$ *is a conformal mapping, then*

$$k_{\mathbb{H}^2}\big(f(x_1), f(x_2)\big) \leq k_{\mathbb{H}^2}(x_1, x_2),$$

for all $x_1, x_2 \in \mathbb{H}^2$.

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Gehring and Osgood proved that quasihyperbolic metric is quasi-invariant under any K-quasiconformal mappings of a domain $D \subset \mathbb{R}^n$.

The result of **Gehring** & **Osgood** can be stated as follows:

Theorem

There exists a constant $c = c(n, K)$ *with the following property: if* f *is a* K*-quasiconformal mapping of domain* D *onto* D'*, then*

$$k_{D'}\big(f(x_1), f(x_2)\big) \leq c \ \max\left(k_D(x_1, x_2), k_D(x_1, x_2)^{\alpha}\right),$$

for all $x_1, x_2 \in D$*, where* $\alpha = K^{1/(1-n)}$*.*

■ Gehring, F. W. & Osgood, B. G., *Uniform domains and the quasi-hyperbolic metric.* J. Analyse Math., **36** (1979), 50–74.

In this talk we shall give a general result for metric spaces.

Theorem

Let X be a c-quasiconvex complete metric space and let Y be a c'-quasiconvex metric space. Suppose that $G \subsetneq X$ and $G' \subsetneq Y$ are two domains and $f : G \to G'$ is an H-quasisymmetry. Then there exists a non-decreasing function $\psi : (0,\infty) \to (0,\infty)$ such that, for all $x, y \in G$,

$$k_{G'}\big(f(x), f(y)\big) \le \psi\big(k_G(x,y)\big).$$

Note that the function $\psi = \psi_{c,c',H}$ and $\psi(t) \to 0$ as $t \to 0$.

It is clear that the converse to the above Theorem is also an interesting problem.

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Here we study the above problem and give a partial answer. That is, for any two c-convex and complete metric spaces, we prove that **quasi-invariance** of the quasihyperbolic metrics implies the corresponding map is **quasiconformal**.

Theorem

Let X be a c-quasiconvex, complete metric space and $G \subsetneq X$ be a domain. Let $G' \subsetneq Y$ be be a domain in a complete metric space Y. Suppose that $f : G \to G'$ is a homeomorphism.

If there is an increasing function $\varphi : (0, \infty) \to (0, \infty)$, and for any sub-domain $E \subseteq G$ and $\forall x, y \in E$,

$$k_{f(E)}\Big(f(x), f(y)\Big) \leq \varphi\Big(k_E(x, y)\Big), \tag{6}$$

then f is an H-quasiconformal mapping with

$$H = e^{\varphi(2c)} - 1.$$

As an application of the above Theorems to the composition map, we obtain

Theorem

Let X (resp. Y) be a c_1 (resp. c_2)-quasiconvex and complete metric space and let Z be a c_3-quasiconvex metric space. For any two domains $G' \subsetneq Y$ and $G'' \subsetneq Z$, if $f : G \to G'$ is an H_1-quasisymmetric mapping and $g : G' \to G''$ is an H_2-quasisymmetric mapping, then $g \circ f$ is an $H = H(c_i, H_i)$-quasiconformal mapping.

■ Xiaojun Huang & L, Quasihyperbolic metric and Quasisymmetric mappings in metric spaces, Trans. A.M.S. 367 (2015), no. 9, 6225-6246.

Thanks for your attention!