Convergence of Riemannian manifolds with curvature bounded below I

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Introduction

- What does it mean that two Riemannian manifolds are "close"?
- In other words, what is a suitable topology to equip a class of Riemannian manifolds with?
- Does the closure under that topology include spaces that are no longer Riemannian manifolds, but have singularities?

These fundamental questions appear in various circumstances in geometric analysis. In the first talk, I will survey some of the convergence results for classes of manifolds with bound(s) on Ricci curvature or scalar curvature, including Gromov compactness theorem, some aspects of Cheeger-Colding theory, and Sormani-Wenger compactness theorem.

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Curvatures in Riemannian geometry

Let (M, g) be a Riemannian manifold and ∇ its Levi-Civita connection. The **Riemannian curvature tensor** R is a $(1,3)$-tensor defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z$$

for vector fields X, Y, Z.

Let $p \in M$ and $u, v \in T_pM$ be linearly independent tangent vectors. The **sectional curvature** of $\{u, v\}$ is defined to be

$$K(u,v) = \frac{g(R(u,v)v,u)}{g(u,u)g(v,v) - g(u,v)^2}.$$

Space forms: manifolds of constant sectional curvature.

Let e_i be an orthonormal basis of T_pM. The **Ricci curvature** Ric is a 2-tensor defined at p by

$$\mathrm{Ric}(X,Y) = \sum_{i=1}^{n} g(R(e_i,Y)X, e_i).$$

Ric is the trace of the curvature tensor with respect to g.

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Curvatures in Riemannian geometry

The **scalar curvature** R is defined by

$$R = \sum_{i=1}^{n} \mathrm{Ric}(e_i, e_i) = \sum_{i,j=1}^{n} g(R(e_i, e_j)e_j, e_i).$$

Scalar curvature is the trace of Ricci curvature with respect to g.

Gromov-Hausdorff Distance

Definition

The **Gromov-Hausdorff distance** between two metric spaces X, Y is denoted by $d_{GH}(X, Y)$ and defined to be

$$d_{GH}(X, Y) = \inf d_H^Z(\phi_X(X), \phi_Y(Y))$$

where the infimum is taken over all embeddings $\phi_X : X \to Z, \phi_Y : Y \to Z$ into a complete metric space Z, and d_H^Z is the Hausdorff distance in Z.

Gromov-Hausdorff Distance

The following definition is an extension of tangent space at a point of a Riemannian manifold to a metric space.

Definition

Let X be a metric space.

1. Let $p \in X$. A pointed metric space (Y, q) is a **tangent cone** at p of X if there exists a sequence $r_i \to 0$ such that

$$\lim_{i \to \infty} r_i^{-1} d_{GH} \left(B_p(r_i) \subset X, B_q(r_i) \subset Y \right) = 0.$$

2. A pointed metric space (Y, q) is a **tangent cone** at infinity of X if there exists a sequence $r_i \to \infty$ such that

$$\lim_{i \to \infty} r_i^{-1} d_{GH} \left(B_p(r_i) \subset X, B_q(r_i) \subset Y \right) = 0.$$

Gromov-Hausdorff Distance

Gromov Compactness Theorem

If (M_i, g_i) is a sequence of compact Riemannian manifolds such that $\mathrm{Ric}_{g_i} \geq \kappa g_i$ and $\mathrm{diam}(M_i, g_i) \leq L$ for some fixed $\kappa, L \in \mathbb{R}$, then there exists a subsequence which converges in Gromov-Hausdorff topology to a metric space.

Convergence for sectional curvature: Alexandrov spaces

Let $k \in \mathbb{R}$. Denote by M_k the 2-dimensional simply connected space form of curvature k.
Let X be a length space, i.e. a metric space such that for any $p, q \in X$, there exists a continuous path γ from p to q that realizes the distance between p and q, so that $|\gamma| = d(p, q)$.

Definition

X is **Alexandrov space** of curvature $\geq k$ if for any triangle Δpqr in X, there exists a comparison triangle $\Delta\bar{p}\bar{q}\bar{r}$ in M_k such that $d(p, q) = d_{M_k}(\bar{p}, \bar{q})$, $d(q, r) = d_{M_k}(\bar{q}, \bar{r})$, $d(r, p) = d_{M_k}(\bar{r}, \bar{p})$, so that for any $s \in qr$ and $\bar{s} \in \bar{q}\bar{r}$ with $d(q, s) = d_{M_k}(\bar{q}, \bar{s})$,

$$d(p, s) \geq d_{M_k}(\bar{p}, \bar{s}).$$

The class of Alexandrov spaces of curvature $\geq k$ is closed with respect to Gromov-Hausdorff convergence.

Convergence for sectional curvature: Alexandrov spaces

Theorem (convergence for sectional curvature)

If (M_i, g_i) is a sequence of Riemannian manifolds of dimension n with sectional curvature $\geq k$ that converges in Gromov-Hausdorff topology to a metric space X, then X is a locally compact Alexandrov space of curvature $\geq k$.

(However, not all locally compact Alexandrov spaces are limits of Riemannian manifolds.)

Alexandrov spaces of curvature $\geq k$ what arise as limit of manifolds with sectional curvature $\geq k$ share many properties with Riemannian manifolds, for example:

- One can define the tangent space at each point, i.e. the "space of angles", and that notion coincides with the Gromov-Hausdorff tangent cone at that point. The tangent cone of an Alexandrov space of Hausdorff dimension n is an Alexandrov space of nonnegative curvature of Hausdorff dimension n. Moreover, every point has a neighborhood homeomorphic to its tangent space.

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Convergence for sectional curvature: Alexandrov spaces

- On Alexandrov spaces of curvature bounded below and Hausdorff dimension 2, one can prove a generalization of Gauss-Bonnet theorem.
- One can define parallel transport along a minimizing path, harmonic maps, etc.

Theorem (splitting theorem for Alexandrov spaces)

If a locally compact Alexandrov space X of nonnegative curvature contains a line, i.e. a minimizing path defined on all of $\mathbb{R}$, then X is isometric to a metric product $\mathbb{R} \times Y$ for some nonnegatively curved Alexandrov space Y.

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Cheeger-Colding theory

Cheeger-Colding theory contains information on the structure of and convergence to Ricci limit spaces.

In geometry, a “rigidity theorem” says that if a geometric quantity such as volume or diameter attains the possible maximal value for the given curvature conditions, then the metric is a certain warped product.

Theorem (Cheeger-Gromoll splitting theorem)

Let (M, g) be a complete Riemannian manifold of dimension n of nonnegative Ricci curvature. If M contains a line, then M is isometric to a product $\mathbb{R} \times N$ for some Riemannian manifold N of dimension $(n-1)$ of nonnegative Ricci curvature.

Cheeger-Colding theory

An "almost rigidity theorem" says that if the quantity is close to being maximal, the metric is close (in a suitable distance) to a warped product.

Let (X, d_X) be an arbitrary metric space, and $f : (a, b) \to \mathbb{R}$ a positive function. The warped product $(a, b) \times_f X$ is the topological space $(a, b) \times X$ equipped with the metric d' given by

$$d'((r_1, x_1), (r_2, x_2)) = \inf \int_0^{d_X(x_1,x_2)} \left(r'(t)^2 + f(r(t))^2\right)^{1/2} dt,$$

where the infimum is taken over continuous paths $r : [0, d_X(x_1, x_2)] \to [r_1, r_2]$ with $r(0) = r_1, r(d_X(x_1, x_2)) = r_2$.

Suppose that $x_1, x_2 \in X$ and $r_1 < r_2$, $r_3 < r_4$. Then $d'((r_2, x_1), (r_4, x_2))$ is determined only by $r_1, r_2, r_3, r_4, d_X(x_1, x_2)$, that is, for some function Q,

$$d'((r_2, x_1), (r_4, x_2)) = Q(r_1, r_2, r_3, r_4, d'((r_1, x_1), (r_3, x_2))). \quad (1)$$

Cheeger-Colding theory

Define $\mathcal{F}$ by

$$\mathcal{F}(r) = -\int_r^b f(u)\, du. \tag{2}$$

If X is an $(n-1)$-dimensional Riemannian manifold with metric g_X, then on $(a,b) \times_f X$,

$$\mathrm{Hess}_{\mathcal{F}} = (\mathcal{F}'' \circ \mathcal{F}^{-1}) g_{(a,b)\times_f X}.$$

Conversely, the existence of a function whose Hessian is conformal to the metric implies that the manifold is a warped product.

Let (M, g, p) be a complete pointed Riemannian manifold, r the distance function from p, and $A_{a,b}$ the annulus $A_{a,b} = \{a \leq r \leq b\}$. If there is an "almost warping" function $\tilde{\mathcal{F}}$, then the distance d induced by g on M is close to d'.

Theorem (Cheeger-Colding [2])

Suppose that

$$\mathrm{Ric}_g \geq (n-1)\Lambda. \tag{3}$$

Let $R > 0$ and $\varepsilon > 0$. There exists $\delta = \delta(R, \varepsilon, \Lambda) > 0$ with the following effect.

Cheeger-Colding theory

Theorem (continued) (Cheeger-Colding [2])

Let $0 < a < b < R$. Suppose that there exists $\tilde{\mathcal{F}} : A_{a,b} \to \mathbb{R}$ such that

$$\operatorname{range} \tilde{\mathcal{F}} \subset \operatorname{range} \mathcal{F}, \tag{4}$$

$$|\nabla \tilde{\mathcal{F}} - \nabla \mathcal{F}| < \delta, \tag{5}$$

$$\frac{1}{\operatorname{vol}(A_{a,b})} \int_{A_{a,b}} |\nabla \tilde{\mathcal{F}} - \nabla \mathcal{F}| < \delta, \tag{6}$$

$$\frac{1}{\operatorname{vol}(A_{a,b})} \int_{A_{a,b}} |\operatorname{Hess}_{\tilde{\mathcal{F}}} - (H \circ \tilde{\mathcal{F}}) g| < \delta, \tag{7}$$

where $\mathcal{F}$ is defined on $A_{a,b}$ by $\mathcal{F}(x) = \mathcal{F}(r(x))$ and $H = \mathcal{F}'' \circ \mathcal{F}^{-1}$.

Cheeger-Colding theory

Theorem (continued) (Cheeger-Colding [2])

Let $x_1, x_2, y_1, y_2 \in B_R(p)$ be such that

$$r(y_1) - r(x_1) = d(x_1, y_1), \tag{8}$$

$$r(y_2) - r(x_2) = d(x_2, y_2). \tag{9}$$

Then,

$$|d(y_1, y_2) - Q\left(r(x_1), r(y_1), r(x_2), r(y_2), d(x_1, x_2)\right)| < \varepsilon. \tag{10}$$

Cheeger-Colding theory

How can one find an almost warping function on a given manifold of nonnegative Ricci curvature? If M has large volume growth, then harmonic functions on annuli can be used. The key is finding a good cutoff function ϕ so that $|\nabla\phi|$ and $|\Delta\phi|$ are bounded above.

Theorem (Cheeger-Colding [2])

Let $0 < a, b$ and $0 < \omega < 1$. Suppose that M has nonnegative Ricci curvature,

$$\frac{\mathrm{vol}(A_{a,b})}{\mathrm{Area}(r^{-1}(a))} \geq (1-\omega)\frac{\int_a^b f^{n-1}(r)\,dr}{f^{n-1}(a)}, \tag{11}$$

and

$$(n-1)\frac{f'(a)}{f(a)} \geq \Delta r \quad \textit{on } r^{-1}(a). \tag{12}$$

Then there exists $\tilde{\mathcal{F}}$ defined on $A_{a,b}$ so that the equations (4)–(7) in Theorem 0.1 are satisfied on $A_{a+\varepsilon,b-\varepsilon}$, where ε can arbitrarily small if ω is sufficiently small.

Cheeger-Colding theory

Instead of solving the Dirichlet problem on annuli, one can also work on the whole manifold by considering the Green function defined on all of $M\backslash\{p\}$.

Definition

A smooth symmetric function $G : M \times M\backslash D \to \mathbb{R}$, where D is the diagonal, is said to be a Green function *(for the Laplacian) if it is the fundamental solution of the Laplace equation, that is,*

$$\Delta_y G(x,y) = -n(n-2)\omega_n \delta_x(y),$$

where ω_n is the volume of the unit ball in the n-dimensional Euclidean space.

The normalization is chosen so that $G = r^{2-n}$ on $\mathbb{R}^n$ if $n > 3$. If M is complete then a Green function always exists (Malgrange-Ehrenpreis theorem).

Definition

A complete Riemannian manifold M is said to be nonparabolic if it possesses a positive Green function G for the Laplacian. M is said to be parabolic otherwise.

Cheeger-Colding theory

Nonparabolicity can be characterized in terms of the volume growth.

Theorem (Varopoulos [25])

If M is complete and has nonnegative Ricci curvature, and of dimension greater than 2, then M is nonparabolic if and only if

$$\int_1^\infty \frac{t}{\mathrm{vol}(B_p(t))}\,dt < \infty$$

for some $p \in M$.

It is possible to choose the minimal positive Green function (Li-Tam [17]).

Definition

M is said to have maximal volume growth if $\lim_{r\to\infty} \frac{\mathrm{vol}(B_p(r))}{\omega_n r^n} > 0$.

If $n \geq 3$, then maximal volume growth implies nonparabolicity.

Cheeger-Colding theory

We focus on the convergence at infinity of M to a cone, so we will take the warping function f to be $f(x) = |x|$. Note that if (M, g) is the Euclidean space $\mathbb{R}^n$, then $G = r^{2-n}$. It is very useful to define the function b by

$$b = G^{\frac{1}{2-n}}, \tag{13}$$

so that $b = r$ on $\mathbb{R}^n$.

When the sectional curvature K is nonnegative outside of a compact set D, Li [16] proved that if M is moreover Kähler, then $M \backslash D$ is isometric to a product of a compact Kähler manifold of nonnegative sectional curvature and a connegatively curved Riemann surface with boundary, based on the estimates in Li-Tam [17].

Theorem (Colding-Minicozzi [7])

Let $R > 0$ and $\Omega > 1$. Suppose that M has nonnegative Ricci curvature and maximal volume growth, so that

$$V_M := \lim_{r\to\infty} \frac{\mathrm{vol}(B_p(r))}{\omega_n r^n} > 0. \tag{14}$$

Then there exists $R_0 > 0$ and $\delta > 0$, depending only on Ω, V_M so that whenever $R > R_0$,

$$\sup_{r\in(R,\Omega R)} \left| \frac{b}{r} - V_M^{\frac{1}{n-2}} \right| < \delta, \tag{15}$$

$$\frac{1}{\mathrm{vol}(A_{R,\Omega R})} \int_{A_{R,\Omega R}} \left| |\nabla b| - V_M^{\frac{1}{n-2}} \right| < \delta, \tag{16}$$

$$\frac{1}{\mathrm{vol}(A_{R,\Omega R})} \int_{A_{R,\Omega R}} \left| \mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g \right| < \delta. \tag{17}$$

δ can be arbitrarily small if V_M is sufficiently close to 1.

Cheeger-Colding theory

One can estimate the Gromov-Hausdorff distance between an annulus and a discrete approximation of it by picking out points along the geodesic rays:

Theorem (Cheeger-Colding [2])

Let $\Omega > 1$ and $\varepsilon > 0$. Suppose that (M, g, p) is complete and has nonnegative Ricci curvature and maximal volume growth, and of dimension $n \geq 3$. Then there exists $R_0 = R_0(\varepsilon, \Omega) > 0$ and a compact metric space X so that, whenever $R > R_0$,

$$d_{GH}\left((A_{R,\Omega R}, p), (A^{C(X)}_{R,\Omega R}, o)\right) < \varepsilon, \tag{18}$$

where $C(X) = [0, \infty) \times_r X$ is the cone over X.

The diameter of X can be bounded from above by a factor determined by the volume growth and the decay rate of the integrals of the gradient of b and trace-free Hessian of b^2.

Cheeger-Colding theory

An important corollary of the above theorem is the following "volume cone is metric cone" theorem.

Theorem (Cheeger-Colding [2], [3])

1. *Let* $r > 0$. *Suppose that* (M_i, g_i, p_i) *is a sequence of complete Riemannian manifolds with* $\mathrm{Ric}_{g_i} \geq -(n-1)\Lambda$, $\mathrm{diam}(M_i, g_i) \leq L$, *and* $\mathrm{vol}(B_{p_i}^{M_i}(r)) \geq V$. *Suppose that this sequence converges in Gromov-Hausdorff topology to a metric space* (X, d, p). *Then the tangent cone at* p *is a metric cone: it is isometric to a warped product* $C(X) = [0, \infty) \times_r X$, *where the cross section* X *is a compact metric space.*
2. *Suppose that* (M, g) *is a complete Riemannian manifold with Ricci curvature bounded below and maximal volume growth. Then the tangent cone at infinity of* (M, g) *is a metric cone.*

Cheeger-Colding theory

In light of this theorem, one may consider the distance to the nearest cone.

Definition (Colding [6])

Let (M, g) be a complete manifold. We define Θ_r to be the scale-free distance to the nearest cone, that is,

$$\Theta_r = \inf \frac{d_{GH}\left((B_p^M(r), p), (B_o^{C(X)}(r), o)\right)}{r} \tag{19}$$

where the infimum is taken over all complete metric space X. (Θ_r is allowed to be ∞ depending on (M, g) and r.)

The preceding results give that Θ_r can be bounded by the weighted L^2-integral of the trace-free Hessian on scale r. In fact, the proofs of the theorems above give a precise relationship between Θ_r and the integral:

Cheeger-Colding theory

Theorem (Colding [6], Theorem 4.7, Corollary 4.8)

Let $\varepsilon > 0$*. Let* (M, g, p) *be a pointed complete manifold of dimension* $n \geq 3$ *with nonnegative Ricci curvature and maximal volume growth, so that*

$$\lim_{r \to \infty} \frac{\mathrm{vol}(B_p(r))}{\omega_n r^n} = V_M > 0.$$

Then there exists $C = C(\varepsilon, n, V_M)$ *so that*

$$\Theta_r^{2+\varepsilon} \leq C r^{-n} \int_{b \leq Cr} \left| \mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g \right| \mathrm{dvol}, \tag{20}$$

and

$$\Theta_r^{2+\varepsilon} \leq C r^{-n} \int_{b \leq Cr} \left| \mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g \right|^2 \mathrm{dvol}. \tag{21}$$

Ricci limit spaces

Let X be a noncollapsed Ricci limit space, i.e. $(M_i^n, g_i, p_i) \to (X, d, p)$ in Gromov-Hausdorff topology where $\mathrm{Ric}_{M_i^n} \geq n-1$ and $\mathrm{vol}(B_1(p_i)) > v > 0$.

[3] The volume measure of (M_i^n, g_i) converges to a unique measure on X which is absolutely continuous with respect to the Hausdorff measure.

A point in X is called **regular** if any tangent cone at p is isometric to $\mathbb{R}^n$; otherwise it is called **singular**. Let $\mathcal{R}$ be the regular set and $\mathcal{S}$ the singular set. Even for points in $\mathcal{S}$, every tangent cone is a metric cone. Define the k**-th stratum** $\mathcal{S}^k$ to be the set of points where tangent cones split in at most k directions, that is,

$$\mathcal{S}^k(X) = \{x \in X : \text{ no tangent cone at } x \text{ is isometric to } \mathbb{R}^{k+1} \times C(Z)\}.$$

Then X is a stratified space: $\mathcal{S}^0 \subset \mathcal{S}^1 \subset \cdots \subset \mathcal{S}^{n-2} = \mathcal{S} \subset \mathcal{S}^{n-1} = X$, so that $\mathcal{S}^{n-1} \backslash \mathcal{S}^{n-2} = \mathcal{R}$. Moreover, $\dim \mathcal{S}^k \leq k$, where dimension is the Hausdorff dimension.

Ricci limit spaces

Theorem (codimension 4 conjecture [4])

If moreover $|\mathrm{Ric}_{M_i^n}| \leq n-1$, then $\dim(\mathcal{S}) \leq n-4$. Here the dimension can be taken to be Hausdorff or Minkowski dimension. Equivalently, $\mathcal{S}^{n-4} = \mathcal{S}^{n-3} = \mathcal{S}^{n-2} = \mathcal{S}$.

On a n-dimensional manifold of Ricci curvature bounded below, $\mathrm{Ric} \geq (n-1)k$, Bishop-Gromov volume monotonicity says that the volume ratio $\frac{\mathrm{vol}(B_p(r))}{B^{M_k}(r)}$ is monotone nonincreasing in r.

The limit space has nonnegative Ricci curvature in the sense that the volume monotonicity holds on the limit space when one replaces volume with the limit measure of Cheeger-Colding.

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Thank you!