Convergence of Riemannian manifolds with curvature bounded below II

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1. Convergence for scalar curvature bounded below

What can we say about convergence of manifolds with a lower bound on scalar curvature only? In this case, we do not have Gromov's compactness theorem anymore.

Sormani and Wenger [25] defined the **intrinsic flat distance** between two compact manifolds, using the measure-theoretic notions of integral currents and weak convergence of metric spaces by Ambrosio-Kirchheim.

Let (X, d, T) be an integer rectifiable current space, meaning that (X, d) is a countably $\mathcal{H}^m$ rectifiable space and T is a linear functional on the space of compactly supported "differential forms" of degree m on X, so that on each Lipschitz chart, T has integer multiplicities. One can define the mass $M(T)$ of the current T.

An oriented Riemannian manifold (M^m, g) of finite volume is naturally an integral current space, with T given by $T(\omega) = \int_M \omega$ for an m-form ω. The mass of T is then given by $M(T) = \mathrm{vol}(M)$.

One can define the boundary of T in a manner so that Stokes' theorem holds.

Intrinsic flat distance

Let Z be a metric space and T_1 and T_2 be two m-integral currents on Z. the **flat distance** between T_1 and T_2 is defined to be

$$d_{\mathcal{F}}^{Z}(T_1, T_2) = \inf\{M(B^{m+1}) + M(A^m) : T_1 - T_2 = A + \partial B\}.$$

The **intrinsic flat distance** between two integral current spaces (X_1, d_1, T_1) and (X_2, d_2, T_2) is defined to be

$$d_{\mathcal{F}}((X_1, d_1, T_1), (X_2, d_2, T_2)) = \inf \left\{ d_{\mathcal{F}}^{Z}(\phi_{1\#}T_1, \phi_{2\#}T_2) \mid \phi_i : X_i \to Z \right\}$$

where the infimum is taken over all common complete metric spaces Z and all isometric embeddings $\phi_i : X_i \to Z$, where $\phi_{i_\sharp}$ is the push-forward map on integral currents.

Theorem (compactness for scalar curvature [26])

If (M_j, g_j) is a sequence of closed oriented Riemannian manifolds such that $\mathrm{diam(M_j)} \leq D$ and $\mathrm{vol}(M_j) \leq V$, then a subsequence converges in intrinsic flat topology to a limit. The limit is an integral current space.

Question: If (M_j, g_j) has nonnegative (positive) scalar curvature, does the limit space have "nonnegative (positive) scalar curvature" in some generalized sense?

Theorem (Gromov [10])

Let M^n be a smooth manifold and g_j a sequence of Riemannian metrics with $R_{g_j} \geq c$. If g_j converges in $\mathcal{C}^0$ to a smooth Riemannian metric g_∞, then $R_{g_\infty} \geq c$.

There are many approaches to define weak notions of scalar curvature on singular metric spaces: using dihedral angles on a polyhedra [10], using Ricci flow on $\mathcal{C}^0$ metrics [2], $\cdots$

For the "small infinitesimal volume" sense of nonnegative scalar curvature, there is a counterexample [1] using the "sewing" technique; A sequence of metrics g_j on $\mathbb{S}^3$ can degenerate into $\mathbb{S}^3$ with a curve identified to a single point. At this singular point, it is unlikely that at this point, the limit space has positive scalar curvature. In this example, there are minimal surfaces Σ_j in $(\mathbb{S}^3, g_j)$ such that $\mathrm{Area}(\Sigma_j) \to 0$ as $j \to \infty$.

Conjecture (Gromov-Sormani [11])

Let $\{M_j^3\}$ be a sequence of closed oriented 3-manifolds without boundary satisfying

$$\mathrm{vol}(M_j) \le V, \mathrm{diam}(M_j) \le D, R(M_j) \ge 0, \mathrm{minA}(M_j) \ge A > 0.$$

Here, $\mathrm{minA}(M_j)$ is defined as the infimum of areas of closed embedded minimal surfaces on M_j. Then a subsequence of $\{M_j\}$ converges in the intrinsic flat topology to a limit space M_∞. Moreover, M_∞ has nonnegative generalized scalar curvature and has Euclidean tangent cones almost everywhere.

Theorem (P.-Tian-Wang [22])

Let $\{M_j^3\}$ be a sequence of closed oriented rotationallly symmetric 3-manifolds without boundary satisfying

$$\mathrm{diam}(M_j) \leq D, R(M_j) \geq 0, \mathrm{minA}(M_j) \geq A > 0.$$

Then a subsequence of $\{M_j\}$ converges in the intrinsic flat topology to a limit space (M_∞, g_∞). Moreover, g_∞ is a rotationally symmetric $\mathcal{C}^0$ metric such that the warping function satisfies the nonnegative scalar curvature in distribution sense. (M_∞, g_∞) has Euclidean tangent cones almost everywhere.

Positive mass theorem: any n-dimensional complete asymptotically flat manifold of nonnegative scalar curvature has nonnegative ADM mass. If ADM mass is 0, then the manifold is $\mathbb{R}^n$.

Question: If the ADM mass is small, is the manifold "close" to $\mathbb{R}^n$?

Theorem (Stability of positive mass theorem [17])

For any $\varepsilon, D, A_0 > 0$ there exists $\delta = \delta(\varepsilon, D, A) > 0$ so that, if M is an n-dimensional complete rotationally symmetric manifold of nonnegative scalar curvature with no closed interior minimal hypersurfaces, such that either M has no boundary or ∂M is a stable minimal hypersurface, with

$$m_{\mathrm{ADM}}(M) < \delta,$$

then

$$d_{\mathcal{F}}\left(T_D(\Sigma_0) \subset M, T_D(\Sigma_0) \subset \mathbb{R}^n\right) < \varepsilon.$$

Here Σ_0 is the $(n-1)$-dimensional sphere or area A_0, and T_D denoted the D-tubular neighborhood.

Penrose inequality: If (M, g) si an n-dimensional complete asymptotically flat manifold of nonnegative scalar curvature whose boundary is an outermost minimal hypersurface, then $m_{\mathrm{ADM}} \geq \frac{1}{2}\left(\frac{\mathrm{Area}(\partial M)}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}}$, where ω_{n-1} is the area of the unit $(n-1)$-sphere. If equality holds, then (M, g) is isometric to a Schwarzschild manifold.

Question: If the ADM mass is small, is the manifold "close" to a Schwarzschild space?

[16] Yes, at least in the rotationally symmetric case: the manifold is "close" to a Schwarzschild space in a certain Lipschitz sense.

2. Uniqueness/non-uniqueness of tangent cone

Let (M, g) be a complete non-compact manifold with nonnegative Ricci curvature and maximal volume growth. Then the tangent cones of (M, g) are metric cones by Cheeger-Colding theory.

In general, tangent cones may depend on the choice of rescalings $\{r_i\}$; one might see different cones at different scales.

- Perelman [23], Cheeger-Colding [4]: examples that have non-isometric tangent cones, constructed by finding multiply warped products with "oscillating" warping functions.
- Colding-Naber [8]: examples where the tangent cones are not only non-isomorphic, but non-homeomorphic.
- What if not only nonnegative Ricci curvature, but Ricci-flat? Hattori [15]: example of a 4-dimensional hyperKähler manifold (with less than maximal volume growth) that has infinitely many non-isometric tangent cones at infinity.

There are works proving uniqueness of tangent cone under additional assumptions on the curvature, geometry of a tangent cone, algebraicity, etc.

- Cheeger-Tian [5] showed that the tangent cone is unique if (M, g) is Ricci-flat and has maximal volume growth, and a tangent cone is smooth and integrable (which is an assumption on the deformation of the Einstein equation on the cross section).
- If one considers Kähler manifolds, Donaldson-Sun [9] proved uniqueness of tangent cone at singularities of a limit of Kähler-Einstein manifolds, by heavily using the algebraic structure of the limit space.

Most relevant to this talk:

Theorem (Colding-Minicozzi [7], Theorem 0.2)

Let M^n be a complete non-compact Ricci-flat manifold of maximal volume growth. If one tangent cone at infinity has smooth cross section, then the tangent cone is unique.

3. Identification of scales

Theorem (Colding-Minicozzi [7])

Suppose that one tangent cone of (M^n, g) is smooth, i.e. it is in the form of $(\mathbb{R}_{\geq 0} \times N, dr^2 + r^2 g_N)$ for some smooth compact Riemannian manifold (N^{n-1}, g_N). Let $\mathcal{A}$ be the set of $\mathcal{C}^{2,\beta}$ metrics g and positive $C^{2,\beta}$ functions w on N, and define a subspace $\mathcal{A}_1$ of $\mathcal{A}$ by

$$\mathcal{A}_1 = \left\{ (g, w) \in \mathcal{A} \,\middle|\, \int_N w \, d\mu_g = \mathrm{Vol}(\partial B_1(0)) \right\}.$$

Define the functional $\mathcal{R} : \mathcal{A}_1 \to \mathbb{R}$ by

$$\mathcal{R} = \frac{1}{2-n} \left(A - \frac{r^{3-n}}{n-2} \int_{b=r} R_{b=r} |\nabla b| \right).$$

Then $\mathcal{R}$ satisfies the Łojasiewicz-Simon inequality for some $\alpha < 1$

$$|\mathcal{R}(g, w) - \mathcal{R}(g_N, V_M)|^{2-\alpha} \leq |\nabla \mathcal{R}|^2 (g, w).$$

The Łojasiewicz inequality implies the critical decay

$$\int_{b\geq r} b^{-n}\left|\mathrm{Hess}^g_{b^2}-\frac{\Delta b^2}{n}g\right|^2_g \leq \frac{C}{\log r^{1+\beta}}. \tag{1}$$

Denote by $\Phi : M\times\mathbb{R}\to M$ the gradient flow of b^2 so that

$$\frac{d\Phi_t(x)}{dt}=\nabla b^2(x).$$

In the special case of the Euclidean space $(M,g,p)=(\mathbb{R}^n,g_{\mathrm{Euc}},0)$, Φ is simply a dilation map $\Phi_t(x)=e^{2t}x$. If we perform a coordinate change to the metric $g_{\mathrm{Euc}}=dr^2+r^2g_{\mathbb{S}^{n-1}}$ by $s=\log r$, so that $r^{-2}g_{\mathrm{Euc}}=ds^2+g_{\mathbb{S}^{n-1}}$ is now a cylindrical metric, then since $r(\Phi_t(x))=e^{2t}x$, it follows that

$$(r\circ\Phi_t)^{-2}\Phi_t^*g_{\mathrm{Euc}}=g_{\mathrm{Euc}},$$

so the metric $(r\circ\Phi_t)^{-2}\Phi_t^*g_{\mathrm{Euc}}$ is constant in t.

Question: a natural identification between scales?

For large $r_0>0$ fixed and $A>0$, would (some version of) $\|(r\circ\Phi_t)^{-2}\Phi_t^*g-(r\circ\Phi_{At})^{-2}\Phi_{At}^*g\|_{C^\infty(g|_{\partial B_p(r_0)})}\leq\psi(t)$, and $\lim_{t\to\infty}\psi(t)=0$, hold? If so, Φ provides a good way of identifying scales.

Theorem (P. [20])

Let (M^n, g) be a complete Ricci-flat manifold of dimension $n \geq 3$ and maximal volume growth. Define the family of metrics $g(t)$ by

$$g(t) = (b \circ \Phi_t)^{-2} \Phi_t^* g. \tag{2}$$

Suppose that a tangent cone at infinity of M has smooth cross section. Let $A > 0$. Then there exist constants $r_0, \beta > 0$ so that for any $r > r_0$ and $T > t > 0$,

$$\fint_{b=r} \left\{ \sup_{v \neq 0} \left| \log \frac{g(At)(v,v)}{g(t)(v,v)} \right| \right\} d\sigma \leq C t^{-\frac{\beta}{2}}, \tag{3}$$

where r_0, β depend on (M, g) only, and C depends on (M, g) and A.

Remark

The idea of conformally changing a given metric by a suitable factor of a positive Green function of a linear elliptic operator has appeared in other works. For instance Schoen used the Green function of the operator $Lu = \Delta u - u$ *in his solution to the Yamabe problem [24].*

Remark

With the aid of some functional inequalities, there is no need to consider the conformally changed metric $g(t)$*. For instance, suppose that the following Hardy-Sobolev type inequality holds on* M*,*

$$\int_{b\leq r} b^{-n}|f-\bar{f}_r|^2 \leq C\int_{b\leq r} b^{2-n}|\nabla f|^2 \qquad \text{for any } f\in\mathcal{C}^\infty(M),\ \lim_{r\to\infty} f = 0. \tag{4}$$

Then one can directly work with the rescaled metrics $\tilde{g}(t) = e^{-4V_M^{\frac{2}{n-2}}t}\Phi_t^* g$ *and obtain the same convergence of* $\tilde{g}(t)$*. The proof carries through using of the Hardy-Sobolev inequality applied to* $f = |\nabla b|^2 - V_M^{\frac{2}{n-2}}$*. One can conclude that the above theorem holds with* $g(t)$ *replaced with* $\tilde{g}(t)$*.*

3. Matrix Harnack inequality

The Li-Yau gradient estimate for heat equation:

Theorem (Li-Yau [18])

Let (M, g) be a compact Riemannian manifold and $f > 0$ a positive solution to the heat equation, $\partial_t f = \Delta f$. Suppose that M has nonnegative Ricci curvature. Then for any $t > 0$ and any vector field V on M, we have

$$\partial_t f + \frac{n}{2t} f + 2Df(V) + f|V|^2 \geq 0. \tag{5}$$

Later this was generalized to a matrix estimate by Hamilton:

Theorem (Hamilton [12])

Suppose moreover that M has parallel Ricci curvature and nonnegative sectional curvature. Then for any $t > 0$ and for any vector field V_i on M, we have

$$D_i D_j f + \frac{1}{2t} f g_{ij} + D_i f \cdot V_j + D_j f \cdot V_i + f V_i V_j \geq 0. \tag{6}$$

Taking the trace of this matrix inequality yields the Li-Yau gradient estimate, although the curvature assumptions are stronger for the matrix inequality. Matrix estimates have also been developed for other situations, such as the heat equation on Kähler manifolds with nonnegative holomorphic bisectional curvature by Ni-Cao [3], Ricci flow by Hamilton [14], Kähler-Ricci flow by Ni [19], and mean curvature flow by Hamilton [13].

Theorem (Colding [6])

If (M^n, g) is nonparabolic and has nonnegative Ricci curvature, then

$$|\nabla b| \leq 1.$$

If equality holds at any point on $M \backslash \{p\}$, then $(M, g) \cong (\mathbb{R}^n, g_{\text{Euc}})$. In addition,

$$\lim_{r \to \infty} \sup_{M \backslash B_r(p)} |\nabla b| = V_M^{1/(n-2)}.$$

On $\mathbb{R}^n$, $b^2 = r^2$ and $\mathrm{Hess}_{b^2} = 2g$. Hence it is natural to ask whether $\mathrm{Hess}_{b^2} \leq 2g$ in general, or at least:

if $\mathrm{Hess}_{b^2} \leq Cg$ near p for some C, then does this bound on Hess_{b^2} extend to M?

Theorem (P. [21])

Suppose that (M^n, g) is nonparabolic and satisfies the following two curvature conditions:

- $R(\nabla b, V, \nabla b, V) \geq 0$ for any vector field V on M,
- $|\nabla \mathrm{Ric}| \leq \epsilon$ for some $\epsilon > 0$.

Suppose that

- $\mathrm{Hess}_{b^2} \leq Dg$ on $M \backslash \{p\}$ for some D, and
- $\mathrm{Hess}_{b^2} \leq Cg$ near p for some $C \leq C(n, \epsilon)$.

Then $\mathrm{Hess}_{b^2} \leq Cg$ on $M \backslash \{p\}$.

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Thank you!