Deformations of Kähler-Einstein metrics and Kähler-Ricci flows

Sungmin Yoo (IBS-Center for Geometry and Physics)

The 16th KIAS Winter School on Geometry

2021.01.26

Table of Contents

Let X: compact Kähler manifold

Theorem (Aubin/Yau 1978)

If $c_1(X) < 0$, then $\exists! \omega_{KE}$ s.t. $\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}$

Let X: general type and fix $\omega_0 \in -c_1(X)$.

$\partial\overline{\partial}$-lemma $\Rightarrow \mathrm{Ric}(\omega_0) + \omega_0 = i\partial\overline{\partial}F$ for some $F \in C^\infty(X)$.

Definition (Elliptic MA eqn)

Find a solution $\varphi \in C^\infty(X)$ satisfying

$$\begin{cases} (\omega_0 + i\partial\overline{\partial}\varphi)^n = e^{\varphi+F}\omega_0^n \\ \omega_0 + i\partial\overline{\partial}\varphi > 0. \end{cases}$$

Then $\omega_{KE} = \omega_0 + i\partial\overline{\partial}\varphi$.

Theorem (Cao, 1985)

For all $t > 0$, there exists a family of Kähler metrics $\omega(t)$ on X satisfying

$$\begin{cases} \frac{\partial}{\partial t}\omega(t) = -\mathrm{Ric}(\omega(t)) - \omega(t), \\ \omega(t)|_{t=0} = \omega_0. \end{cases}$$

Moreover, $\lim_{t\to\infty} \omega(t) = \omega_{KE}$. This is called the Kähler-Ricci flow (KRF).

Definition (Parabolic MA eqn)

Find a solution $\varphi \in C^\infty(X \times [0,\infty))$ satisfying

$$\begin{cases} (\omega_0 + i\partial\overline{\partial}\varphi)^n = e^{\frac{\partial}{\partial t}\varphi + \varphi + F}\omega_0^n, \\ \varphi(t)|_{t=0} = 0. \end{cases}$$

Define $\omega(t) := \omega_0 + i\partial\overline{\partial}\varphi(t)$.

Setting

Let $\pi : \mathcal{X}^{n+1} \to Y^1$ be a family of canonically polarized mfds, i.e.,

- $\pi : \mathcal{X} \to Y$: *proper holomorphic surjective submersion*
- $X_y := \pi^{-1}(y)$: *general type, i.e.,* $c_1(X_y) < 0$

Theorem (Ehresmann's fibration theorem)

Let $\pi : \mathcal{X} \to Y$: proper surjective submersion

Then $X_y \cong X_{y'}$ (diffeomorphic) for all $y, y' \in Y$

Let ω: d-closed $(1,1)$-form on $\mathcal{X}$ s.t. $\omega_y := \omega|_{X_y} > 0$ and $[\omega_y] = -c_1(X_y)$.

$\Rightarrow \omega_y^{KE} = \omega_y + i\partial\overline{\partial}\psi_y$: KE metric on X_y (Aubin/Yau)

$\omega_y(t) = \omega_y + i\partial\overline{\partial}\varphi_y(t)$: KRF on X_y (Cao)

$\Rightarrow \exists\ \rho := \omega + i\partial\overline{\partial}\psi$: d-closed $(1,1)$-form on $\mathcal{X}$ s.t. $\rho|_{X_y} = \omega_y^{KE}$

$\exists\ \omega(t) := \omega + i\partial\overline{\partial}\varphi$: d-closed $(1,1)$-form on $\mathcal{X}$ s.t. $\omega(t)|_{X_y} = \omega_y(t)$

Remark

- ρ *is called the fiberwise KE.*
- $\omega_{WP} = \int_{\mathcal{X}/Y} \rho^{n+1}$ *Weil-Petersson metric.*
- $\omega(t)$ *is called the fiberwise KRF.*
- $\lim_{t\to\infty} \omega(t) = \rho$

Theorem (Schumacher, 2012)

- $\rho \geq 0$ *on* $\mathcal{X}$.
- $\rho > 0$ *on* $\mathcal{X}$ *if* $\pi : \mathcal{X} \to Y$ *is effectively parametrized.*

Theorem (Paun, 2018)

ρ *extends across the singularities of the map* π *as a positive current.*

Theorem (Berman, 2013)

- $\omega(t) \geq 0$ *on* $\mathcal{X}$ *if* $\omega \geq 0$.
- $\omega(t) > 0$ *on* $\mathcal{X}$ *if* $\omega > 0$ *and* $\pi : \mathcal{X} \to Y$ *is effectively parametrized.*

Goal: Study a holomorphic family of non-compact complete Kähler mfds!

Table of Contents

Question: Which non-compact mfd admits a complete KE metric?

Definition (Bounded geometry)

We say that (X,ω) has bounded geometry of order k if for each $p \in X$ there exists a holomorphic chart (U_p, ξ_p) centered at p and constants $r, C, C_k > 0$ independent of p satisfying

(i) $\mathbb{B}_r(0) \subset V_p := \xi_p(U_p) \subset \mathbb{C}^n$,

(ii) $\frac{1}{C}\left(\delta_{\alpha\bar{\beta}}\right) \leq \left(g_{\alpha\bar{\beta}}\right) \leq C\left(\delta_{\alpha\bar{\beta}}\right)$,

(iii) $\left\|g_{\alpha\bar{\beta}}\right\|_{C^k(V_p)} \leq C_k$.

where $\omega = i g_{\alpha\bar{\beta}} d\xi^\alpha \wedge d\overline{\xi^\beta}$ for the coordinates $\xi_p = (\xi^1, \ldots, \xi^n)$.

For any functions $u \in C^\infty(X)$, we define a norm by

$$\|u\|_{k+\epsilon} := \sup_{p \in X}\{\|u \circ \xi_p^{-1}\|_{C^{k+\epsilon}(V_p)}\},$$

$\tilde{C}^{k+\epsilon}(X)$:= Banach completion of $\{u \in C^\infty(X) : \|u\|_{k+\epsilon} < \infty\}$.

Theorem (Cheng-Yau, 1980)

- (X,ω): *complete Kähler mfd with bounded geometry*
- $\mathrm{Ric}(\omega)+\omega = i\partial\bar{\partial}F$ *for some fn* $F\in \tilde{C}^\infty(X)$

$\Longrightarrow \exists!$ *complete KE metric* ω_{KE} *with* $\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}$

Theorem (Chau, 2004)

- (X,ω): *complete Kähler mfd with bounded curvatures*
- $\mathrm{Ric}(\omega)+\omega = i\partial\bar{\partial}F$ *for some smooth bounded fn* F *on* X

$\Longrightarrow \exists$ *Kähler-Ricci flow* $\omega(t)$ *for all* $t>0$ *and* $\lim_{t\to\infty}\omega(t)=\omega_{KE}$

Theorem (Wu-Yau, 2020)

Let (X,ω): *complete Kähler mfd with* $-C_1 < Hol(\omega) < -C_2 < 0$

$\Longrightarrow \exists!$ *complete KE metric* ω_{KE} *with* $\mathrm{Ric}(\omega_{KE}) = -\omega_{KE}$

Definition

A smooth bdd domain $\Omega \subset\subset \mathbb{C}^n$ is called a strongly pseudoconvex domain if $\exists$ a smooth strictly plurisubharmonic defining function r on $\overline{\Omega}$, i.e.,

- $\Omega = \{r(z) < 0\}$, $\partial\Omega = \{r(z) = 0\}$, $dr \neq 0$ on $\partial\Omega$ *(def. fn)*
- $i\partial\overline{\partial} r > 0$ on $\overline{\Omega}$ *(str. psh)*

Remark

- {*strongly convex domains*} $\subset$ {*strongly pseudoconvex domains*}
- *bdd pcx. domain = increasing union of str. pcx. domains*

Proposition (Cheng-Yau)

Let Ω be a bounded strongly peudoconvex domain in $\mathbb{C}^n$.

$\omega := i\partial\overline{\partial}(-\log(-r))$ *has bdd geometry,* $\mathrm{Ric}(\omega) + \omega = i\partial\overline{\partial}F$, $F \in \tilde{C}^\infty(\Omega)$.

Proposition (Cheng-Yau)

Let (X, ω_0) *be a Kähler manifold with* $\mathrm{Ric}(\omega_0) < 0$

Let Ω be a bounded strongly peudoconvex domain in X.

Then $\omega := -\mathrm{Ric}(\omega_0) + i\partial\overline{\partial}(-\log(-r))$ *has bdd geometry and satisfies*

$\mathrm{Ric}(\omega) + \omega = i\partial\overline{\partial}F$ *for some* $F \in \tilde{C}^\infty(\Omega)$.

Remark

In the above case, we have asymptotic boundary estimates of ω_{KE} (Cheng-Yau, Lee-Melrose)

Definition

$\pi : D \to S$ *is a holomorphic family of str. pcx. domains if*

- $\pi : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$: *coordinate projection map* $\pi(z_1, \ldots, z_n, s) = s$
- D: *smooth domain in* $\mathbb{C}^{n+1}$, $\quad S := \pi(D)$
- $D_y := D \cap \pi^{-1}(y)$: *smooth str. pcx. domain in* $\mathbb{C}^n$, $\forall y \in S$

$\Rightarrow \exists\ \omega$: d-closed smooth $(1,1)$-forms on D such that $\omega_y := \omega|_{D_y}$ is a complete Kähler metric with bounded geometry.

$\Rightarrow \exists\ \rho$: fiberwise KE on D

Theorem (Choi, 2015)

- $\rho > 0$ *on* D *if* D *is strongly pseudoconvex in* $\mathbb{C}^{n+1}$.
- $\rho \geq 0$ *on* D *if* D *is pseudoconvex in* $\mathbb{C}^{n+1}$.

Setting

- $\pi : X \to Y$: *holomorphic surjective submersion*
- ω_X: *Kähler form on* X *satisfying* $\mathrm{Ric}(\omega_y) < 0$, *where* $\omega_y := \omega_X|_{X_y}$
- D: C^∞*-domain in* X, $\quad S := \pi(D)$
- $D_y := D \cap \pi^{-1}(y)$: C^∞*-bdd. str. pcx. domain in* X_y, $\forall y \in S$
- $\pi : \overline{D} \to S$: *proper* ($\Rightarrow \overline{D_y} := \overline{D} \cap \pi^{-1}(y)$: *diffeomorphic,* $\forall y \in S$)

Theorem 1 (Choi, Y-)

If D *is strongly pseudoconvex in* X, ρ *is positive definite on* D.

Setting

- $\pi : X \to Y$: *holomorphic surjective*
- W: $\exists$ *analytic set containing all singular values of* π *and* $\pi|_{\partial D}$ *in* S
- *For all regular fibers, the assumptions of Theorem 1 hold.*

Theorem 2 (Choi, Y-)

If D *admits a complete Kähler metric* $\tilde{\omega}_D$ *satisfying* $Scal(\tilde{\omega}_y) > C$*,*

then ρ *extends to* D *as a positive current.*

On the other hand, we also have $\omega(t)$: fiberwise KRF

Theorem 3 (Choi, Y-)

Suppose that $\omega \geq 0$ on D and strictly positive at least one point on each fiber D_y. Then $\omega(t) > 0$ on D for all $t > 0$

Proposition

If D is pseudoconvex in $\mathbb{C}^{n+1}$, there exist a defining function r of D s.t. $\omega := i\partial\overline{\partial}(-\log(-r))$ satisfies the above assumption.

Corollary

- *$\rho \geq 0$ on D if D is pseudoconvex in $\mathbb{C}^{n+1}$.*

Table of Contents

$\pi : \mathcal{X} \to Y$ with $\dim_{\mathbb{C}}(Y) = 1$

τ: a d-closed smooth real $(1,1)$-form on $\mathcal{X}$ s.t. $\tau|_{X_y} > 0$

$v := \frac{\partial}{\partial s} \in T'_y(Y)$

Definition

- v_τ: *horizontal lift of v is a $(1,0)$ vector field satisfying*
  (1) $d\pi(v_\tau) = v$
  (2) $< v_\tau, w >_\tau = 0$ *for* $\forall w \in T'X_y$
- $c(\tau)$: *geodesic curvature of τ is defined by*

$$c(\tau) = < v_\tau, v_\tau >_\tau$$

Remark

- $\tau^{n+1} = c(\tau) \cdot \tau^n \wedge \pi^*(ids \wedge d\bar{s})$
- $\tau \geq 0 \Longleftrightarrow c(\tau) \geq 0 \quad (\tau > 0 \Longleftrightarrow c(\tau) > 0)$

PDE of geodesic curvatures

For each fiber D_y, consider geodesic curvatures $c(\rho)$ and $c(\omega(t))$.

Proposition (Schumacher)

$$(-\Delta + id)\, c(\rho) = \|\overline{\partial} v_\rho\|^2,$$

where Δ is the Laplace-Beltrami operator of ω_y^{KE}.

Proposition (Berman)

$$\left(\frac{\partial}{\partial t} - \Delta_t + id\right) c(\omega(t)) = \|\overline{\partial} v_{\omega(t)}\|^2,$$

where Δ_t is the Laplace-Beltrami operator of $\omega_y(t)$.

Use a non-compact version of elliptic and parabolic maximum principle!

Theorem (Almost Maximum Principle)

Let (X, g) be a complete Riemannian manifold

Let f be a bounded from above smooth function on X.

- *(Omori, 1967) If sectional curvatures are bounded from below, there exists a sequence of points $\{x_k\} \in X$ satisfying*

$$f(x_k) > \limsup f - \frac{1}{k}, \quad |df(x_k)| < \frac{1}{k}, \quad \textit{and} \quad \mathrm{Hess} f(x_k) < \frac{1}{k} g.$$

- *(Yau, 1975) If Ricci curvatures are bounded from below, there exists a sequence of points $\{x_k\} \in X$ satisfying*

$$f(x_k) > \limsup f - \frac{1}{k}, \quad |df(x_k)| < \frac{1}{k}, \quad \textit{and} \quad \Delta f(x_k) < \frac{1}{k}.$$

Let (X, g) be a complete Riemannian manifold with bounded curvatures

Theorem (Weak maximum principle (Shi))

Let f be a smooth bounded function on $X \times [0, T)$ satisfying

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) f \geq 0 \quad \textit{whenever } f \leq 0.$$

If $f \geq 0$ on X at $t = 0$, then $f \geq 0$ on $X \times [0, T)$.

Theorem (Strong maximum principle)

Suppose that $\sup_{X \times [0,T)} f(x, t) \geq 0$ and $f(x, 0) > 0$ at some point $x \in X$.

Then $f(x, t) > 0$ on $X \times [0, T)$.

Theorem (Weak maximum principle (Ni))

Let f be a smooth function on $X \times [0, T)$ satisfying

$$\left(\frac{\partial}{\partial t} - \Delta_t\right) f \geq 0 \quad \textit{whenever } f \leq 0.$$

Assume that $\exists c > 0$ such that $f_- := -\min\{f, 0\}$ satisfies

$$\int_0^T \int_X (-r)^c (f_-)^2 \, dV_t dt < \infty.$$

If $f \geq 0$ on X at $t = 0$, then $f \geq 0$ on $X \times [0, T)$.

Thank you for your attention.