Gauss curvature flow with an obstacle

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Gauss curvature flow (GCF)

Let $X_0 : M^n \to \mathbb{R}^{n+1}$ be a smooth immersion.

Gauss curvature flow If $X : M^n \times [0, T) \to \mathbb{R}^{n+1}$ satisfies

$$\frac{\partial}{\partial t} X(p, t) = -K(p, t)\nu(p, t)$$

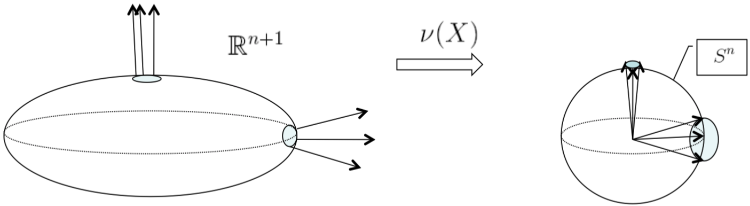
with $X(\cdot, 0) = X_0$, then the one-parameter family of hypersurfaces $\Sigma_t = X(M^n, t)$ is said to be the Gauss curvature flow.
Here, $K(p, t)$ is the Gauss-Kronecker curvature of Σ_t at $X(p, t)$ and $\nu(p, t)$ is the outward unit normal vector to Σ_t at $X(p, t)$.

One of the higher dimensional versions of the curve shortening flow.

Tumbling stone on a beach - phenomena in nature - Firey

Modeling

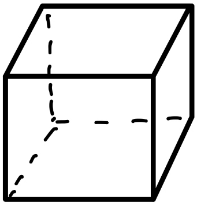
A stone is subject to wear under impact from any random angle by the sea.

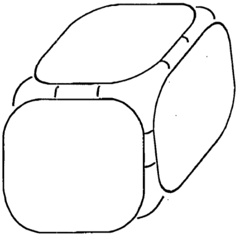


$$K(X) = \lim_{\rho \to 0} \frac{\mathrm{vol}(\nu(B_\rho(X)))}{\mathrm{vol}(B_\rho(X))} \quad \text{(Gauss Curvature)}$$

Erosion is heuristically proportion to the measure of image of the Gauss map.

Example





Theorem (Hamilton '94)

If a hypersurface has flat sides then the flat sides still exist under the Gauss curvature flow for some time.

If the cube evolves under the mean curvature flow, then the flat sides will disappear immediately. (SMP, heat exchange)

Volume change with a constant speed

Under GCF, the volume $V(t)$ enclosed by Σ_t satisfies

$$\frac{\partial}{\partial t}V(t) = -\int_{\Sigma_t} K d\sigma_{\Sigma_t} = -|\mathbb{S}^n|$$

by an analogy of Gauss-Bonnet theorem.

Apply maximum principle to $\frac{\partial}{\partial t}K = K(h^{-1})^{ij}\nabla_i\nabla_j K + K^2 H$

GCF must shrink to a point at $T_* = V(0)/|\mathbb{S}^n|$.

Relation to (L_p-)Minkowski problem

Given a Borel measure $\mu \geq 0$ on $\mathbb{S}^{n-1}$, find a convex body K such that $S_{K,p} = \mu$. (Existence and uniqueness?)

convex body $K \quad \longleftrightarrow \quad$ L_p-surface area measure $S_{K,p} = h_K^{1-p} dS_K$

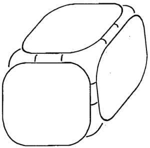
By the Brunn-Minkowski inequality, solutions to (classical) Minkowski problem can be given by

$$\inf \left\{ \int_{\mathbb{S}^{n-1}} h_L(X) d\mu(X), L \in K^n, V(L) \geq 1 \right\}$$

and the uniqueness follows from the equality condition on the Brunn-Minkowski inequality.

The self-similar solution to the Gauss curvature flow is the solution to L_0-Minkowski problem ($p = 0$, so called logarithmic Minkowski problem).

Preservation of flat sides

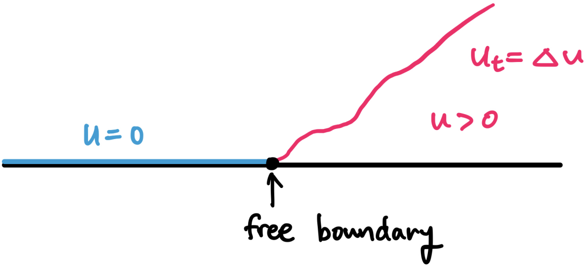


Λ: flat sides. $\Gamma = \partial\Lambda$: free boundary, the interface between flat parts and non-flat parts.

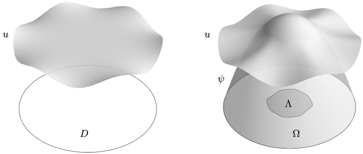
- [Daskalopoulos–Hamilton '99] short time; [Daskalopoulos–Lee '04] long time.
- Chopp–Evans–Ishii showed that if X_0 is smooth, free boundaries does not move during some waiting time period.
- Choi found that the optimal rate of movement is $C^{1,\frac{1}{n-1}}$.

Therefore, regularity near free boundary is key to understand how the stone evolves.

Flat sides = one phase problem



GCF with flat sides can be formulated as obstacle problem ($\psi \equiv 0$).



Membrane with obstacle

Obstacle problem for membrane (variational inequality):

$$\underset{u \geq \psi}{\text{minimize}} \quad \int_D \frac{1}{2} |\nabla u|^2 dx$$

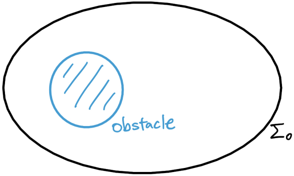
Then $\Delta u = 0$ in $\{u > \psi\}$ and $\Delta u \leq 0$ in D.

Linear operator $\rightarrow$ "flat side" $=$ "obstacle"
Nonlinear operator $\rightarrow$ "flat side" $\neq$ "obstacle"

GCF with flat side $\neq$ GCF with obstacle

Question: What is "obstacle problem" for GCF?

Model for the flow with an obstacle



When a stone have a hard core, the hypersurface cannot penetrate the hard core.

Difference with the flat sides problem:

growing contact set vs shrinking flat sides

Focus on the neighborhood of the free boundary since this part is important as in flat sides problem.

Main results - stationary obstacle

Theorem (Lee–L. '18, $n = 2$, $\alpha \leq 1$, stationary)

Let X_0 and Ψ be smooth strictly convex closed hypersurfaces. Then GCF with an obstacle has a unique solution for $t \in [0, \infty)$ with the optimal $C^{1,1}$-regularity. In particular,

1. $0 \leq \lambda_i \leq C$, λ_i: *principal curvature.*
2. *On the non-coincidence set $X(x, t) \notin \Phi$,*

$$0 < c(d(X, \Lambda)) \leq \lambda_i(X) \leq C,$$

   where $d(X, \Lambda)$ denotes the distance from x to $\{X \in \Phi\}$.

3. *There is a finite time T_* such that $\Sigma_t = \Phi$ for all $t \geq T_*$.*

K becomes zero on the free boundary.

Main results - moving obstacle

Theorem (Lee–L., all n and $\alpha > 0$, varying obstacle)

Let X_0 and Ψ be smooth strictly convex closed hypersurfaces. Assume that Ψ is a slowly contracting obstacle. Then the Gauss curvature flow with an obstacle has a unique solution for $t \in [0, \infty)$ with the optimal $C^{1,1}$-regularity. In particular,

1. *The principal curvatures λ_i satisfy $c(X_0, \Phi) \leq \lambda_i \leq C(X_0, \Phi)$.*
2. *There is a finite time T_* such that $\Sigma_t = \Phi$ for all $t \geq T_*$.*

Free boundary in various fields

Physical phenomena
rolling stones, melting of ice (Stefan problem)

Geometry
collapsed geometry, degenerate solution to Minkowski problem

Economics and Finance
portfolio selection, American option pricing, transaction cost

Thank you!

References I

- Ben Andrews, *Gauss curvature flow: the fate of the rolling stones*, Invent. Math. **138** (1999), no. 1, 151–161. MR 1714339
- Simon Brendle, Kyeongsu Choi, and Panagiota Daskalopoulos, *Asymptotic behavior of flows by powers of the Gaussian curvature*, Acta Math. **219** (2017), no. 1, 1–16. MR 3765656
- D. Chopp, L. C. Evans, and H. Ishii, *Waiting time effects for Gauss curvature flows*, Indiana Univ. Math. J. **48** (1999), no. 1, 311–334. MR 1722203
- P. Daskalopoulos and R. Hamilton, *The free boundary in the Gauss curvature flow with flat sides*, J. Reine Angew. Math. **510** (1999), 187–227. MR 1696096

References II

- P. Daskalopoulos and Ki-Ahm Lee, *Worn stones with flat sides all time regularity of the interface*, Invent. Math. **156** (2004), no. 3, 445–493. MR 2061326
- William J. Firey, *Shapes of worn stones*, Mathematika **21** (1974), 1–11. MR 362045
- Richard S. Hamilton, *Worn stones with flat sides*, A tribute to Ilya Bakelman (College Station, TX, 1993), Discourses Math. Appl., vol. 3, Texas A & M Univ., College Station, TX, 1994, pp. 69–78. MR 1423370
- Arshak Petrosyan, Henrik Shahgholian, and Nina Uraltseva, *Regularity of free boundaries in obstacle-type problems*, Graduate Studies in Mathematics, vol. 136, American Mathematical Society, Providence, RI, 2012. MR 2962060

References III

Kaising Tso, *Deforming a hypersurface by its Gauss-Kronecker curvature*, Comm. Pure Appl. Math. **38** (1985), no. 6, 867–882. MR 812353