Poincaré conjecture

"Every simply connected closed n-manifold is homeomorphic to the n-sphere."

$n=1$; Trivial.

$n=2$; classical result in 19th century.

$n \geq 7$; Smale, 1961

$n=6$; Stallings, 1962

$n=5$; Zeeman, 1961

$n=4$; Freedman, 1982 (→ Fields. 1986)

"Topology"

In 1983, R. Hamilton introduced the Ricci flow

$$\frac{\partial}{\partial t} g_{ij}(\cdot, t) = -2Ric_{ij}(\cdot, t)$$

$$= \Delta_g g_{ij} + \text{lower order terms.}$$

In 2002, Perelman settled the $n=3$ case by using the Ricci flow.

"Geometric analysis"

Smooth Poincaré conjecture

"Every simply connected smooth closed n-manifold is diffeomorphic to a round n-sphere."

$n = 1, 2, 3, 5, 6 \Rightarrow$ True

↳ Ricci flow

There exist counter-examples from $n = 7$.

$\boxed{\begin{array}{l}\#\text{ of (exotic) spheres (c.f. J. Milnor 1956*)} \\ n=7: 28, \quad n=8: 2, \quad n=9: 8, \ldots\end{array}}$

$n = 4$ remians an open problem.

Remark) $\exists$ ∞ many exotic $\mathbb{R}^4$, while $\nexists$ exotic $\mathbb{R}^n$ for $n \neq 4$.

Topological problems in geometric flows.

1. 4D smooth Poincaré conjecture
2. 4D smooth Schoenflies problem.
3. Topological classification of Calabi-Yau threefolds
4. Tightness in contact manifolds

Homeomorphism VS Ambient isotopy.

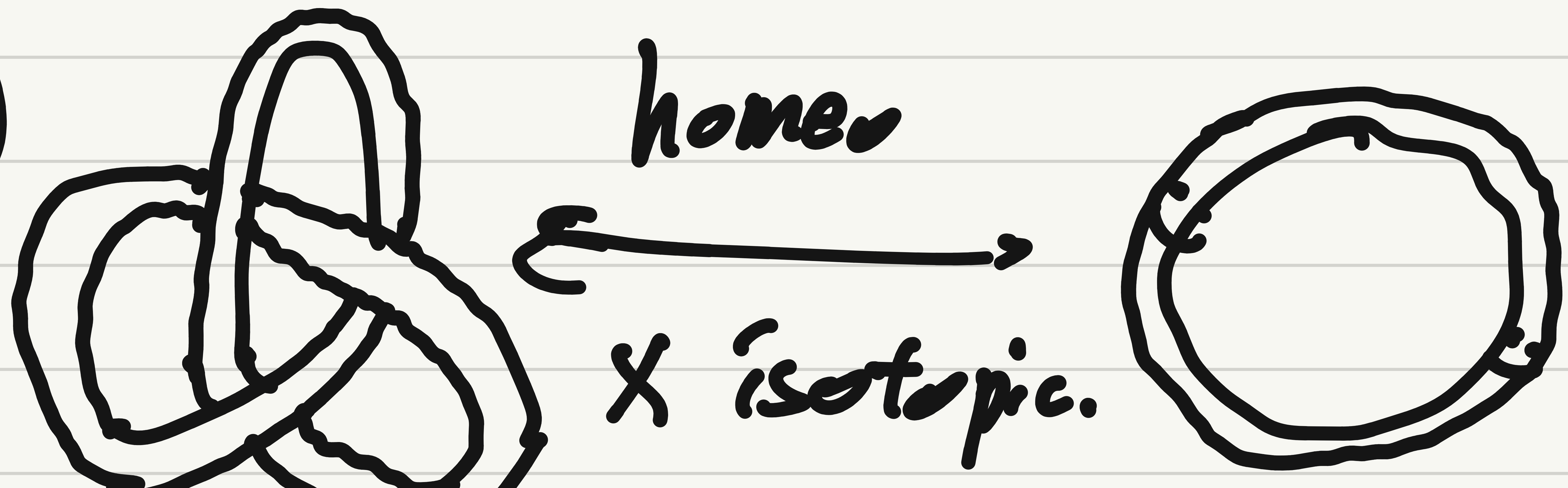
Def. Given topological space X, Y.
a function $f: X \to Y$ is
a homeomorphism if
① f is 1-1,
② " continuous.
③ f^{-1} is " .

Thm: Every closed simple curve in $\mathbb{R}^n$
is homeomorphic to S^1.

Ex)

"Def: $g: N \to M$, $h: N \to M$ are embeddings."
$F: M \times [0,1] \to M$ is continuous.
$F_0 = Id$, F_t is a homeo from M to M
$F_1 \circ g = h$.
$\Rightarrow F$ is an ambient isotopy
taking g to h.

Ex)



Trefoil knot standard torus

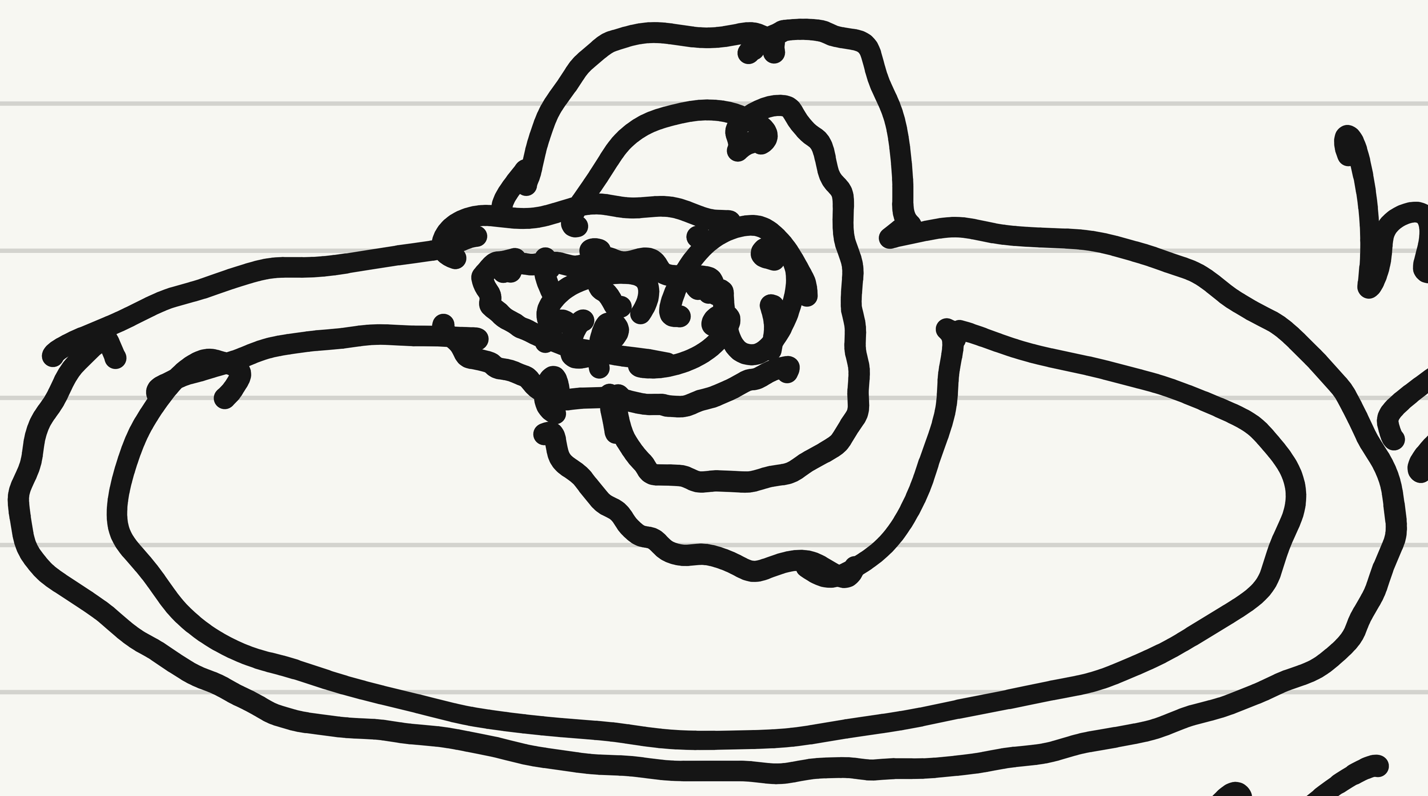
· Jordan - Schoenflies theorem.

Every closed simple curve in $\mathbb{R}^2$ bounds a disk.

(bounds : is the boundary of)

Q. Does every simply - connected closed hypersurface embedded in $\mathbb{R}^n$ bound a ball B^n ?

A. No. Alexander's horned sphere



homeo $\simeq S^n$

X simply connected

Q. Let Σ be a smooth hypersurface embedded in $\mathbb{R}^n$, diffeomorphic to a round S^{n-1}. $\Rightarrow$ Σ is unknotted!?

A. True for all $n \neq 4$.

The case $n=4$ is an open problem.

"4D Smooth Schoenflies problem"

Conjecture ??

Geometric Analysists have been trying to show that the problem is true by using the mean curvature flow.

Let $X: M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth one parameter family of immersions satisfying $\frac{\partial}{\partial t} X = \Delta_g X$.

Then, we call $M_t =: X(M^n, t)$ a mean curvature flow.

c.f. Ricci flow

$$\frac{\partial}{\partial t} g_{ij} = -2Ric_{ij} = \Delta g_{ij} + \text{lower order terms.}$$

Heat equation (Diffusion equation)

Let $I=[a,b] \subset \mathbb{R}$, $T>0$.

$u : I \times [0,T) \to \mathbb{R}$ is a smooth function satisfying

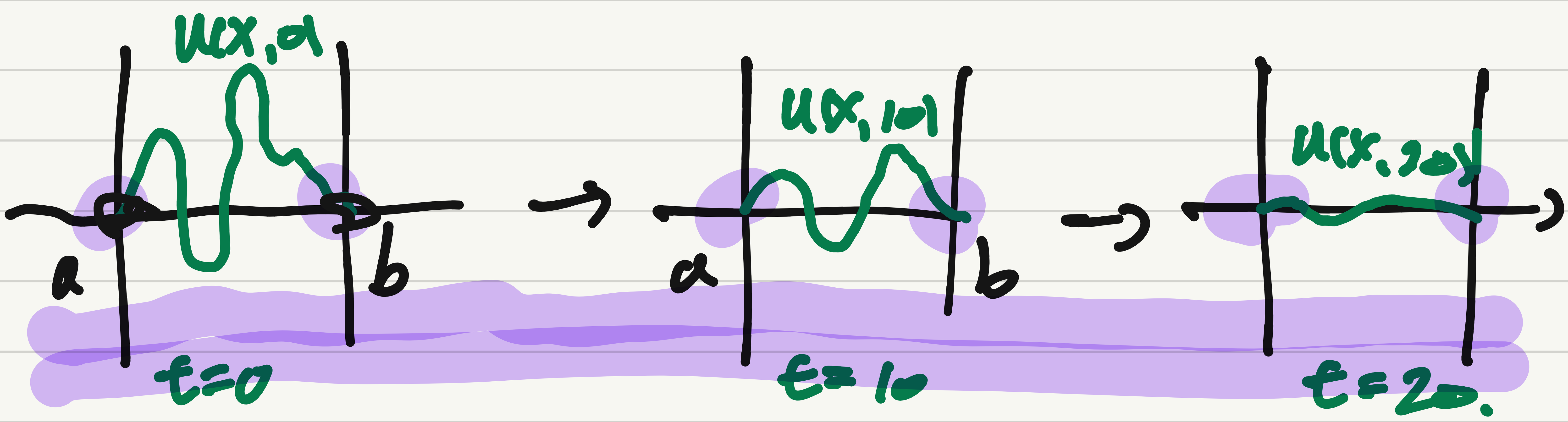
$$\frac{\partial}{\partial t} u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t). \quad \text{—} \ (*)$$

Then, u is a solution to the heat equation $(*)$

$f(t)$ $\qquad$ $g(t)$

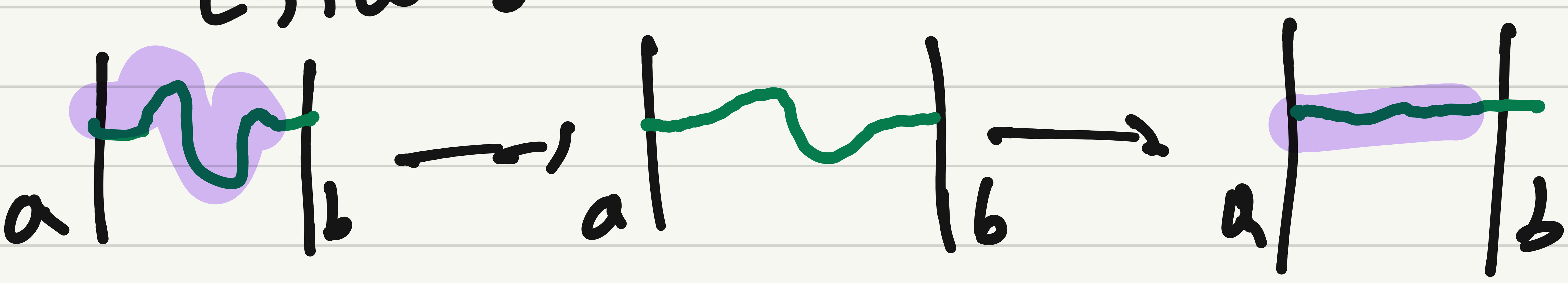
Ex) Suppose $u(a,t) = u(b,t) = 0$ for all $t \geq 0$.

Then. $\displaystyle\lim_{t \to +\infty} \sup_{I} |u(x,t)| = 0.$



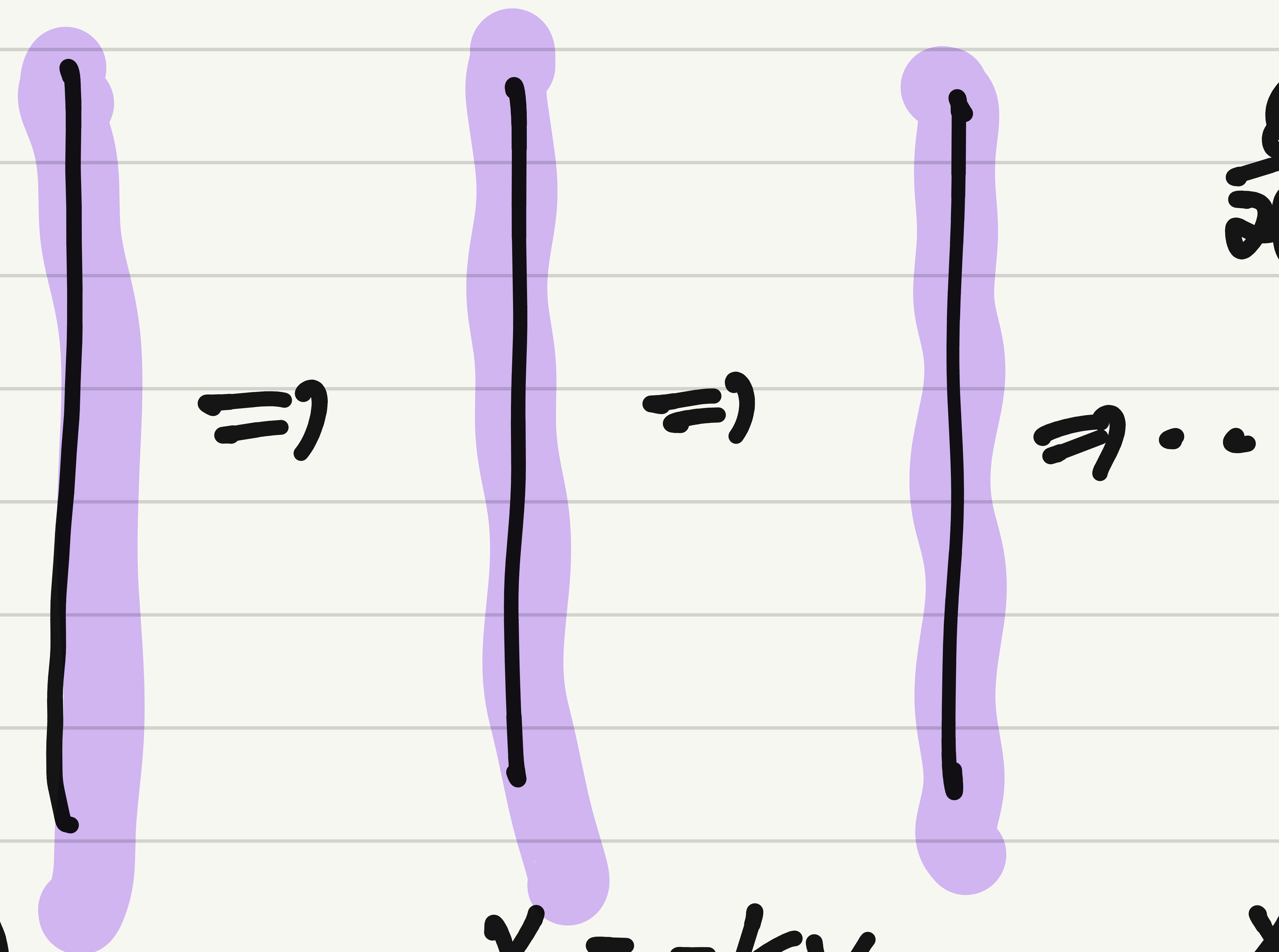
Ex) Suppose $u_x(a,t) = u_x(b,t) = 0 \quad \forall t \geq 0$

$\Rightarrow \displaystyle\lim_{t \to +\infty} \sup_{I} \left| u(x,t) - \fint_a^b u(x,0)\,dx \right| = 0.$



Equilibrium (?) of the curve shortening flow. ($n=1$, MCF in $\mathbb{R}^2$)

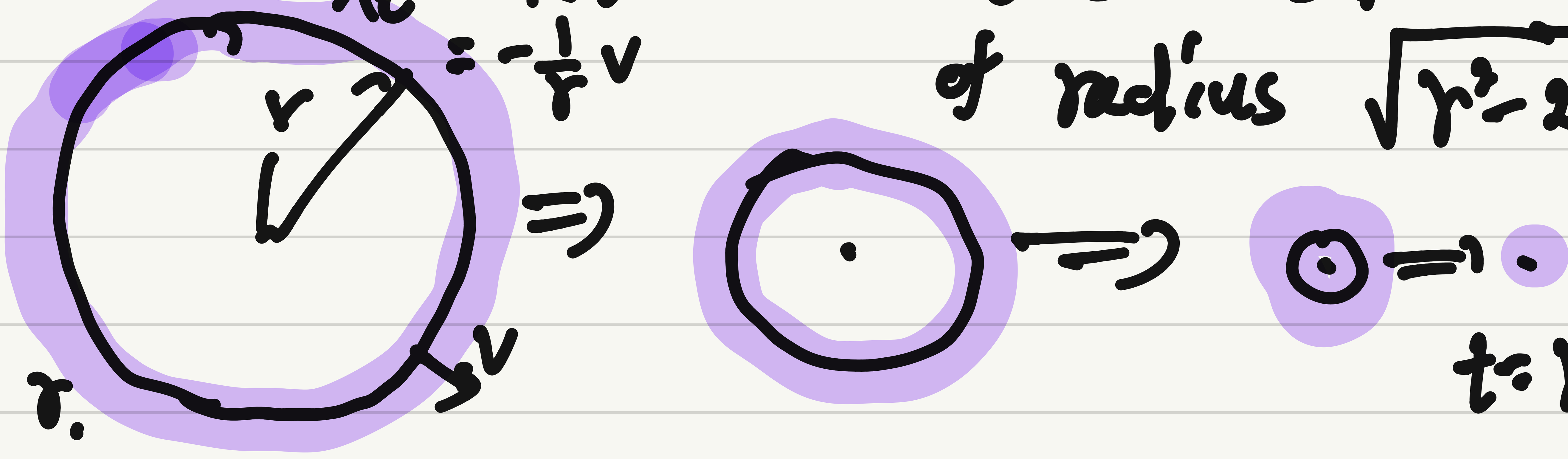
Ex 1) M_0 is a line



$$\frac{\partial}{\partial t}X = \Delta_g X = -k\nu = 0$$

Ex 2) M_0 is a circle of radius r.

$X_t = -k\nu = -\frac{1}{r}\nu$



X_t is a circle of radius $\sqrt{r^2-2t}$.

$t = r^2/2$.

proof) $\rho(t)$ denote the radius

$\Rightarrow \rho' = \langle X_t, \nu \rangle = -k = -1/\rho$

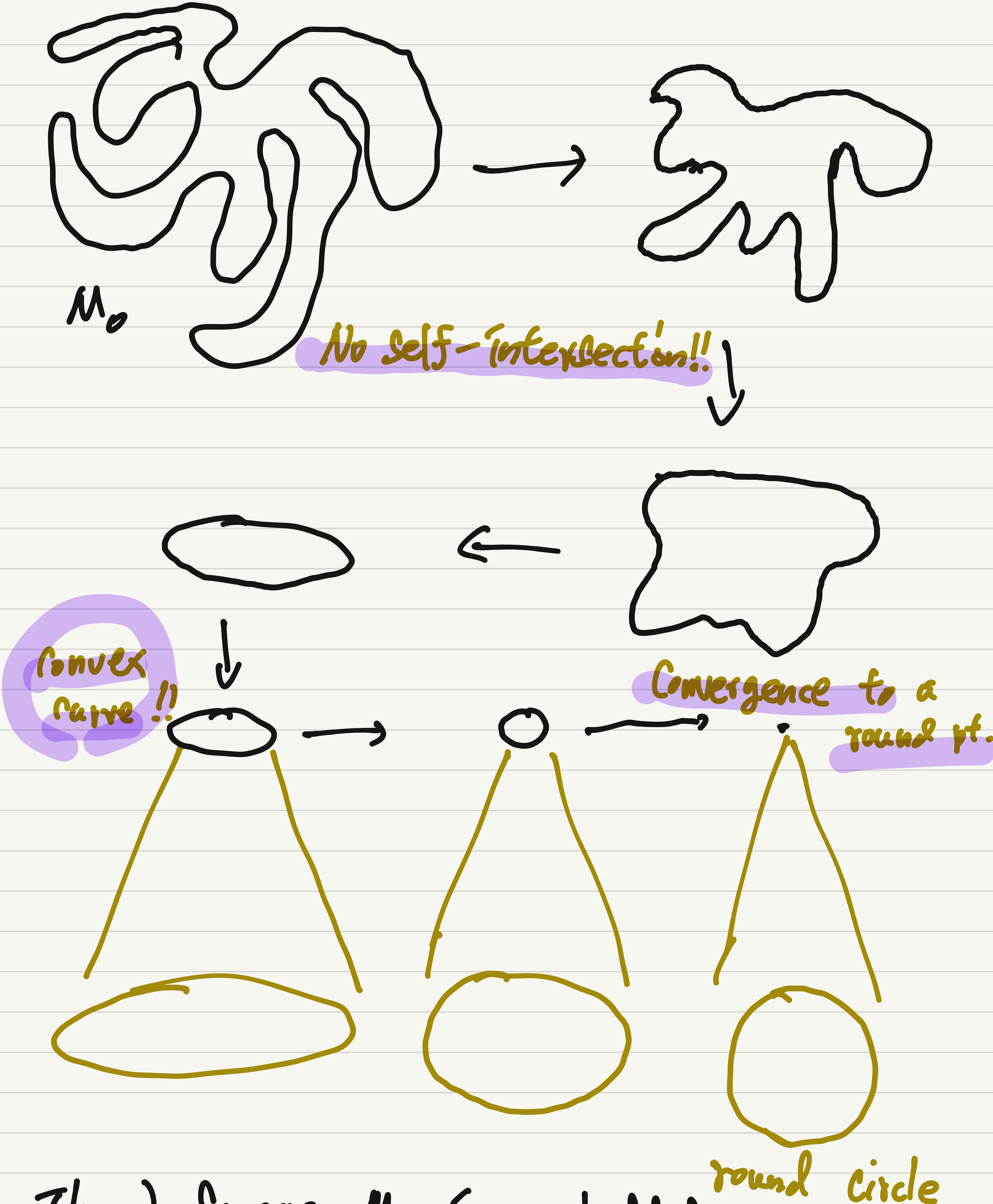
$\Rightarrow -1 = \rho\rho' = \frac{1}{2}(\rho^2)'$

$\Rightarrow \rho^2(t) = \rho^2(0) - 2t = r^2 - 2t$

$\therefore \rho(t) = \sqrt{r^2-2t}$.

Thm) Let M_0 be a closed simple curve in $\mathbb{R}^2$

Then, $\exists\, T < +\infty$ such that M_t converges to a round point as $t \to T$



Thm) Suppose M_0 is embedded.
Then, M_t remains embedded
while the MCF is smooth.