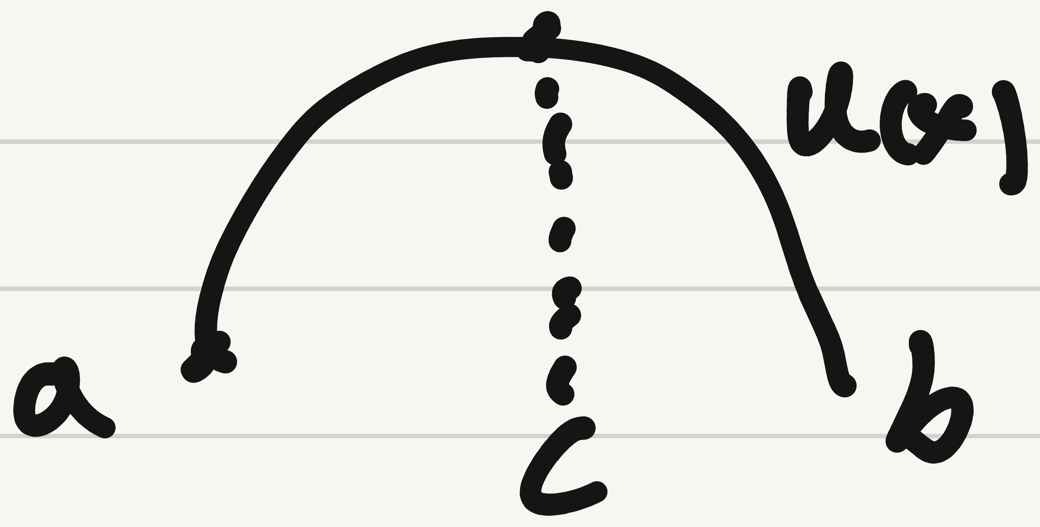
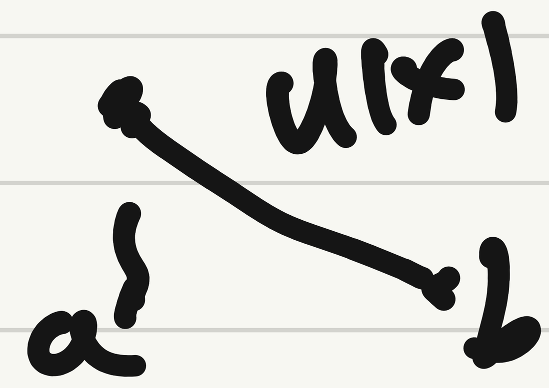
# Maximum Principle

Let $u : [a,b] \to \mathbb{R}$ is of class $C^2$.

Then $\exists\, c \in [a,b]$ s.t. $u(c) = \max\limits_{[a,b]} u$.

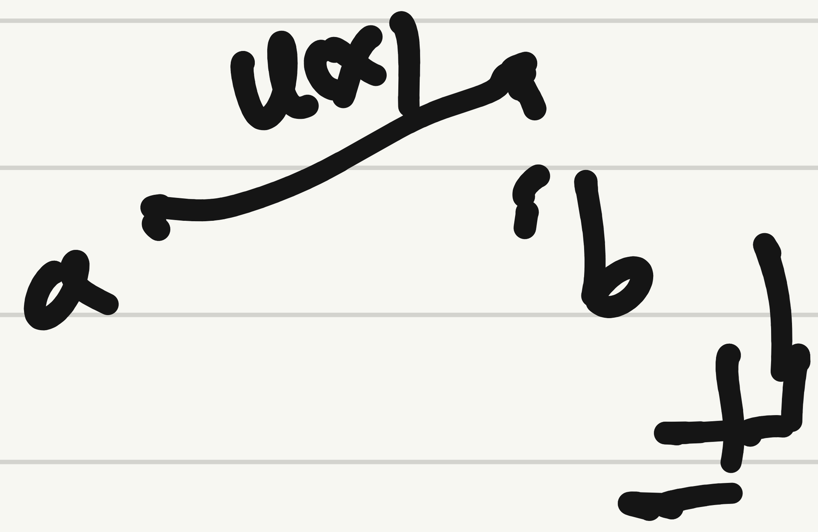
If $a < c < b$, then $u''(c) \le 0$, $u'(c) = 0$.




If $a = c$, then $u'(c) \leq 0$

If $a = b$, then $u'(c) \geq 0$




## Application.

Ex 1) Suppose $u(a) = u(b) = 0$

$$u'' + u' - u = 0 \quad \text{in } [a,b]$$

Then, $u(x) \leq 0 \quad \forall x \in [a,b]$

proof) Suppose $u(c) = \max u > 0$

Then. $c \neq a$, $c \neq b$.

$\Rightarrow u''(c) \le 0$, $u'(c) = 0$.

$\Rightarrow u(c) = u''(c) + u'(c) \le 0$ ※

$\therefore u \leq 0$ in $[a,b]$

Ex 2) Let $u : [a,b] \times [0,T] \to \mathbb{R}$

$u(a,t) \le 0, \quad u(b,t) \le 0 \quad \forall t \in [0,T]$

$u(x,0) \le 0, \quad \forall x \in [a,b]$

$u_t = u_{xx} - u_x - u \quad \text{in } [a,b] \times [0,T]$

Then, $u(x,t) \le 0 \quad \forall (x,t) \in [a,b] \times [0,T]$

proof) Suppose $u(x_0,t_0) = \max u$.

Then, $x_0 \in (a,b)$, $t_0 > 0$.

Hence, $u_{xx}(x_0,t_0) \le 0$, $u_x(x_0,t_0) = 0$

In addition, $u_t(x_0,t_0) \ge 0$

$(\because u(x_0,t) \le u(x_0,t_0) \quad \forall t \le t_0)$

$\Rightarrow u(x_0,t_0) = u_{xx} - u_x - u_t \le 0$

※.

$\therefore u(x,t) \le 0 \quad \text{in } [a,b] \times [0,T]$ □

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"I have made a living off the maximum principle."

Reference: "PDEs" by L. Evans

"$2^{nd}$ order parabolic DEs" by G. Lieberman.

"Elliptic PDEs" by Q. Han & F. Lin

"Elliptic PDEs of $2^{nd}$ order" by D. Gilbarg – N. Trudinger.

"Lectures on MCFs" by Xi-Ping Zhu.

Lemma) **Weak maximum principle.**

Let $\Omega \subset \mathbb{R}^n$ be an open set.

$u : \overline{\Omega} \times [0,T] \to \mathbb{R}$ satisfies

$$\frac{\partial}{\partial t} u(x,t) \leq a_{ij}(x,t) \frac{\partial^2}{\partial x_i \partial x_j} u(x,t) + b_i(x,t) \frac{\partial}{\partial x_i} u(x,t)$$

where $a_{ij} \xi_i \xi_j \geq \lambda |\xi|^2$, $(\lambda > 0)$

holds for all $\xi \in \mathbb{R}^n$.

If $u \leq 0$ on $\partial\Omega \times [0,T]$

and $\Omega \times \{0\}$.

Then, $u \leq 0$ holds in $\overline{\Omega} \times [0,T]$

proof in $n=1$). Define $u^\varepsilon = u - \varepsilon t$, $\varepsilon < 0$.

$\Rightarrow u^\varepsilon_t = u_t - \varepsilon = a u^\varepsilon_{xx} + b u^\varepsilon_x - \varepsilon$

Suppose $u^\varepsilon(x_0, t_0) = \max u^\varepsilon > 0$

$\Rightarrow u^\varepsilon_{xx} \leq 0$. $u^\varepsilon_x = 0$. $u^\varepsilon_t \geq 0$

$\Rightarrow \varepsilon = a u^\varepsilon_{xx} + b u^\varepsilon_x - u^\varepsilon_t \leq 0$ ※

$\therefore u^\varepsilon \leq 0$ in $\overline{\Omega} \times [0,T]$ $\forall \varepsilon > 0$

Passing $\varepsilon \to 0$ yields $u \leq 0$.

Thm) Strong maximum principle

If $u(x_0, t_0) = 0$

at some $(x_0, t_0) \in \Omega \times [0,T]$

then, $u = 0$ in $\Omega \times [0,T]$.

pf)

Let $M_t = (x, u(x,t)) \in \mathbb{R}^{n+1}$

be a MCF, namely $X_t = -H\nu$.

Recall. $X = (x, u(x,t)) \Rightarrow X_t = (0, u_t)$

$$H = \operatorname{div}\left( Du / \sqrt{1+|Du|^2} \right)$$

$$\nu = (Du, -1) / \sqrt{1+|Du|^2}$$

$$\Rightarrow \frac{u_t}{\sqrt{1+|Du|^2}} = -\langle X_t, \nu \rangle$$

$$= H = \operatorname{div}\left( \frac{Du}{\sqrt{1+|Du|^2}} \right)$$

$$\therefore u_t = \sqrt{1+|Du|^2} \operatorname{div}\left( Du / \sqrt{1+|Du|^2} \right)$$

$$= \left( \delta_{ij} - \frac{u_i u_j}{1+|Du|^2} \right) u_{ij} \quad \text{--- } (*)$$

1D) $u_t = u'' / (1+|u'|^2)$

Ex) $M_t = (x, u(x,t))$, $N_t = (x, v(x,t))$

Solve $(*)$ in $\Omega \times [0,T]$

and $u \le v$ on $\partial\Omega \times [0,T]$

and $\Omega \times \{0\}$.

Then, $u \le v$ in $\Omega \times [0,T]$

If $u = v$ at some pt in $\Omega \times (0,T]$

then $u = v$ in $\Omega \times [0,T]$

proof (in 1D) Define $w = u - v$.

Then, $w \le 0$ if $x \in \partial\Omega$ or $t = 0$.

$$w_t = u_t - v_t = \frac{u''}{1+|u'|^2} - \frac{v''}{1+|v'|^2}$$

$$= \frac{(u-v)''}{1+|u'|^2} + v''\left(\frac{1}{1+|u'|^2} - \frac{1}{1+|v'|^2}\right)$$

$$= \frac{w''}{1+|u'|^2} + v''\frac{|v'|^2 - |u'|^2}{(1+|u'|^2)(1+|v'|^2)}$$

$$= \frac{w''}{1+|u'|^2} - \frac{v''(v'+u')w'}{(1+|u'|^2)(1+|v'|^2)}.$$

$\square$.

Thm) $M_t \subset \mathbb{R}^{n+1}$ is a smooth closed MCF for $t \in [0,T)$.

and $M_0$ is embedded.

Then, each $M_t$ is embedded.

proof) Suppose NOT.

Then, $(p_0, t_0)$ and $(p_1, t_0)$ s.t.

$M_t$ is embedded for $t < t_0$,

$X(p_0, t_0) = X(p_1, t_0)$, $p_0 \neq p_1$.

Since $M_t$ is $C^\infty$,

$$\sup_{0 \le t \le t_0} \sup_{M_t} |A|^2 \le C < +\infty.$$

We rotate $M_t$ so that $\nu(p_0, t_0) = -e_{n+1}$.

and translate to have $X(p_0, t_0) = 0$.

$\Rightarrow \exists\ \delta > 0$ and $u : B_\delta(0) \times [t_0 - \delta, t_0] \to \mathbb{R}$

$v :$ " $\to \mathbb{R}$

s.t. $(x, u(x,t)), (x, v(x,t)) \in M_t$ in $B_\delta \times [t_0 - \delta, t_0]$

and $u \le v$ in $B_\delta \times [t_0 - \delta, t_0]$

$u = v$ at $(0, t_0)$

Thus. by the strong max prin.

$u = v$ in $B_\delta(0) \times [t_0 - \delta, t_0]$

$\Rightarrow$ $M_t$ is NOT embedded for $t \in [t_0 - \delta, t_0]$

$\Rightarrow\!\!\!\!\!/$ . !?

Remark) Suppose $M_0$ is a Knotted $C^\infty$ hypersurface embedded in $\mathbb{R}^{n+1}$.

Then. $M_t$ remains Knotted until $M_t$ lose the smoothness.

($\Leftrightarrow$ until $M_t$ develops singularities)

where $|A| = +\infty$.

Thm) $M_0^1$ is embedded and encloses $M_0^2$.

$\Rightarrow$ $M_t^1$ encloses $M_t^2$ while they are $C^\infty$.

Thm) A closed MCF $M_t$ must develop a singularity in finite time.

proof) $\exists R \gg 1$ s.t. $M_0 \subset B_R(0)$

Let $N_t = \{ x \in \mathbb{R}^{n+1} \mid |x| = R^2 - 2t \}$.

Then. $N_t$ is a $C^\infty$ MCF for $t \in [0, R^2/2)$

and $N_0$ encloses $M_0$.

Suppose $M_t$ is $C^\infty$ for $t \in [0, R^2/2]$

Then. $M_t$ is enclosed by $N_t$ for $t \in [0, R^2/2)$.

However, $N_t \to \{0\}$ as $t \to R^2/2$.

$\times$.

Ex) $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}$ solves

$$u_t = (1+|Du|^2)^{1/2} \operatorname{div}\left(Du/\sqrt{1+|Du|^2}\right)$$

and $u(x,0) = 0 \quad \forall x \in \mathbb{R}^n$.

Then. $u(x,t) = 0 \quad$ in $\mathbb{R}^n \times [0,T)$

Remark) $\exists\, u \not\equiv 0$ s.t.

$$u_t = u_{xx} \quad \text{in } \mathbb{R} \times [0,T)$$

and $u(x,0) = 0 \quad \forall x \in \mathbb{R}$.

See Tychonoff's counter-example of the uniqueness.

Thm) $M_0$ is a closed strictly convex simple curve in $\mathbb{R}^2$.

Then. $M_t$ remains closed convex and simple,

proof) Let $k$ denote the curvature w/ $k(\cdot,0) > 0$.

By Zhu's note, $k_t = k_{ss} + k^3$

By max. prin. $\Rightarrow k \geq \min k(\cdot, 0) > 0$.

$\Rightarrow$ convex.

Thm) $M_0$ is a closed strictly convex hypersurface embedded in $\mathbb{R}^n$.
$\Rightarrow$ $M_t$ remains convex.

pf) Let $k_1, \cdots, k_n$ denote the principal curvatures of $M_t$.

Then. $K := k_1 \cdots k_n = \det(h_{ij}) \det(g^{ij})$

where $g$ is the metric and $h$ is the $2^{nd}$ fundamental form.

By Zhu's lecture notes

$$\partial_t g_{ij} = -2H h_{ij}, \quad (H = g^{ij} h_{ij})$$

$$\partial_t h_{ij} = \Delta_g h_{ij} - 2H h_{ik} g^{kl} h_{jl} + |A|^2 h_{ij}$$

where $\Delta_g = g^{ij} \partial_i \partial_j + g^{ij} \Gamma_{ij}^k \partial_k$.

$$\Rightarrow \partial_t K^{\frac{1}{n}} = \Delta_g K^{\frac{1}{n}} - \left( \frac{\partial K^{\frac{1}{n}}}{\partial h_{ij} \partial h_{kl}} \right) \nabla_m h_{ij} \nabla^m h_{kl} + \frac{1}{n} K^{\frac{1}{n}} H$$

$$\geq \Delta_g K^{\frac{1}{n}} + \frac{1}{n} K^{\frac{1}{n}} H.$$

$\Rightarrow$ $K \geq \min_{M_0} K > 0$. $\Rightarrow$ every $k_i > 0$

$\Rightarrow$ Convex !!

Given $X(p,t) \in M_t$, we denote the inscribed radius of $M_t$ at $X(p,t)$ by $R(p,t)$. Namely

If $H(p,t)R(p,t) \geq \alpha > 0$ in $M_t$, then we say $M_t$ is $\alpha$-noncollapsed.

Thm) If $M_0$ is closed embedded and $\alpha$-noncollapsed, then $M_t$ remains $\alpha$-noncollapsed.

proof) Recall $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$.

Define $Z : M^n \times M^n \times [0,T) \to \mathbb{R}$ by

$$Z(p,q,t) = \frac{1}{2} H(p,t) \| X(p,t) - X(q,t) \|^2 - \alpha \langle X(p,t) - X(q,t), \nu(p,t) \rangle$$

$Z(\cdot,\cdot,t) \geq 0 \iff M_t$ is $\alpha$-noncollapsed

By max prin. $Z(\cdot,\cdot,0) \geq 0 \Rightarrow Z(\cdot,\cdot,t) \geq 0$

See B. Andrews, Geom & Topo. 2012.