"Mean convex flow and Schoenflies problem."

Let $\kappa_1 \leq \cdots \leq \kappa_n$ denote the principal curvatures of a hypersurface Σ^n in $\mathbb{R}^{n+1}$

If $H = \kappa_1 + \cdots + \kappa_n > 0$, we say that Σ^n is mean convex.

If $\kappa_1 + \kappa_2 > 0$, we say that Σ^n is 2-convex

Thm) An embedded 2-convex sphere $\Sigma^n \subset \mathbb{R}^{n+1}$ bounds B^{n+1}.

(i.e. unknotted, contractable, (ambient) isotopic to round S^n)

See Huisken-Sinestrari 09'. Brendle-Huisken 16'. Haslhofer-Kleiner 17'

Buzano-Haslhofer-Hershkovits 16'.

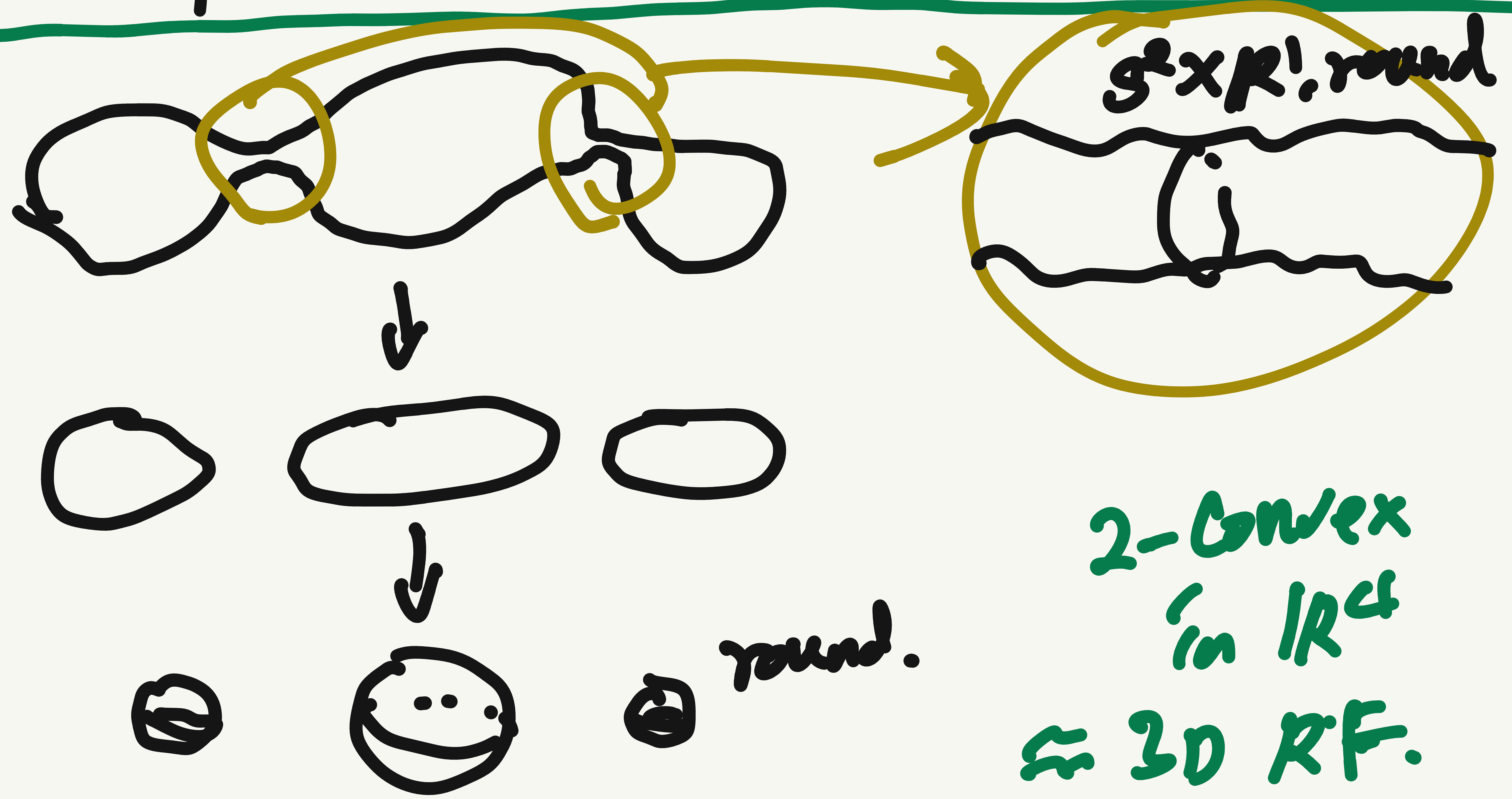
$\approx$ Perelman 02'. (3D Ricci flow)

Remark) · 2-convex in $\mathbb{R}^3$ $\Longleftrightarrow$ mean convex in $\mathbb{R}^3$.

· Mean convex flow in $\mathbb{R}^4$ $\Longleftarrow$ 4D Ricci flow.
Technically...

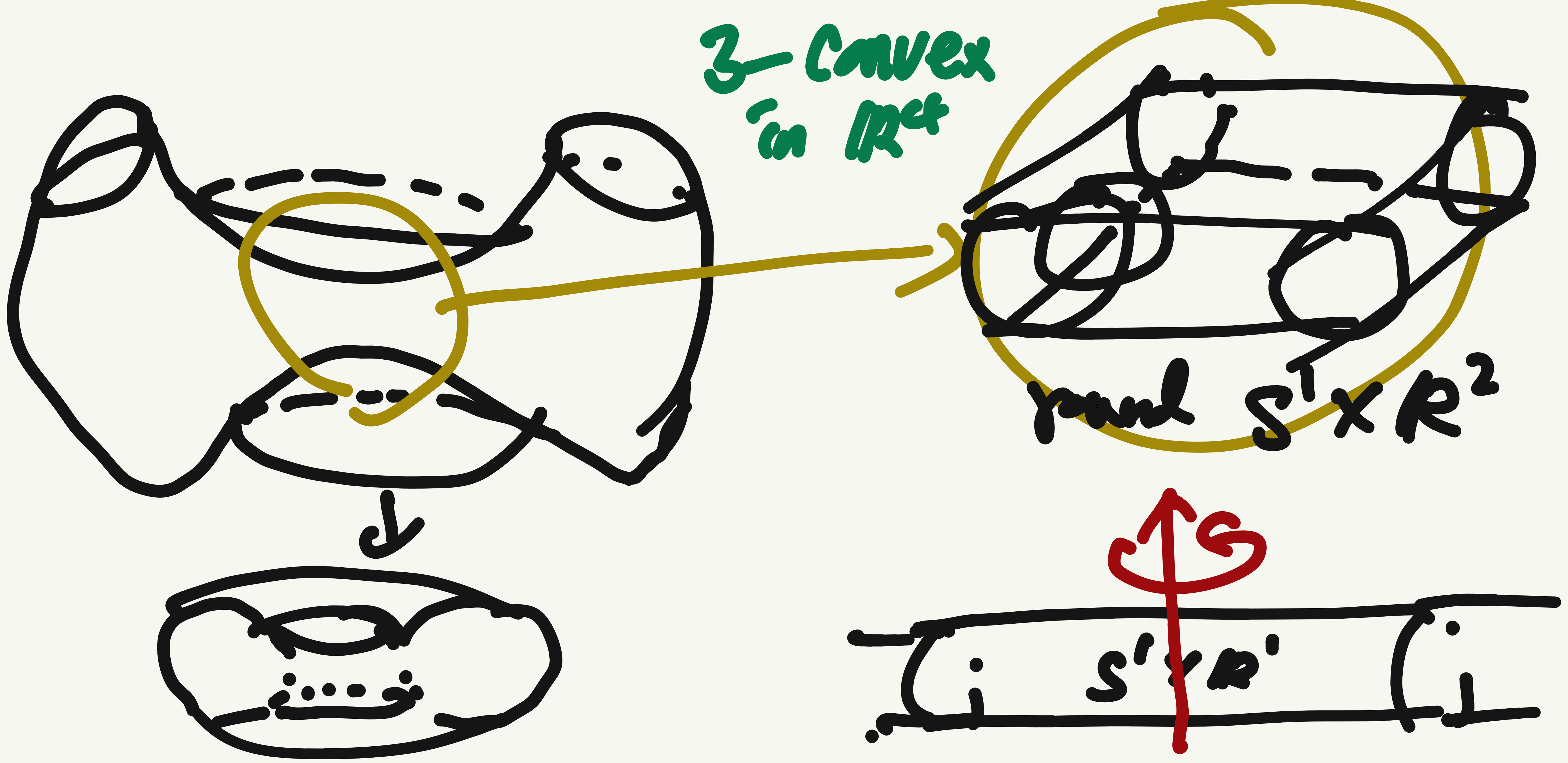
Q. Every embedded smooth mean convex 3-sphere in $\mathbb{R}^4$ bounds B^4 ??

Snapshot of 2-convex flows and 3-convex flows $(k_1+k_2+k_3>0)$



2-Convex in $\mathbb{R}^4$
$\Leftarrow$ 3D RF.

3-convex in $\mathbb{R}^4$



Huisken's monotonicity [90'] Given $X_0 = (x_0, t_0) \in \mathbb{R}^{n+1} \times [0, T)$

$$\frac{d}{dt}\int_{M_t} \rho_{X_0}\, dg = -\int_{M_t} \left\| H\nu - \frac{x - x_0}{2(t - t_0)} \right\|^2 \rho_{X_0}\, dg \le 0, \quad \text{for } t < t_0$$

where $\rho_{X_0}(x,t) = [4\pi(t_0 - t)]^{-\frac{n}{2}} e^{-\frac{|x - x_0|^2}{4(t_0 - t)}}$.

We consider rescaling $\tau = -\log(t_0 - t)$ and .
$\tilde{M}_\tau = (t_0 - t)^{-1/2}(M_t - x_0)$

$$\Rightarrow \frac{d}{d\tau}\int_{\tilde{M}_\tau} \tilde{\rho}\, dg = -\int_{\tilde{M}_\tau} \left\| H\nu - \frac{1}{2}x \right\|^2 \rho\, dg, \quad \text{for } \tau < +\infty$$

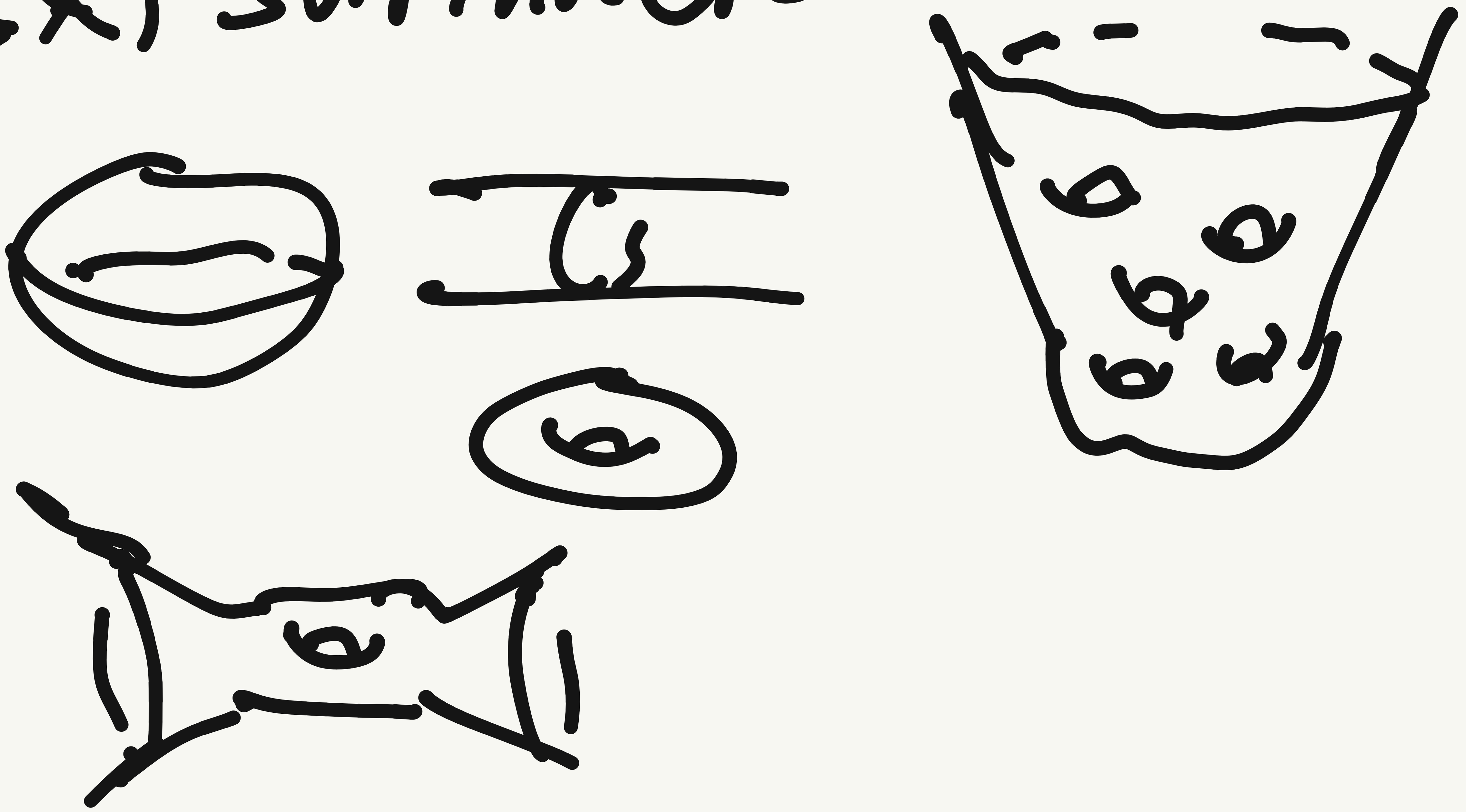
where $\tilde{\rho}(x) = (4\pi)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4}}$. Hence. $H\nu - \frac{1}{2}x \to 0$ as $\tau \to +\infty$

Def) We say that Σ is a shrinker if $H = \frac{1}{2}\langle x, \nu \rangle$.

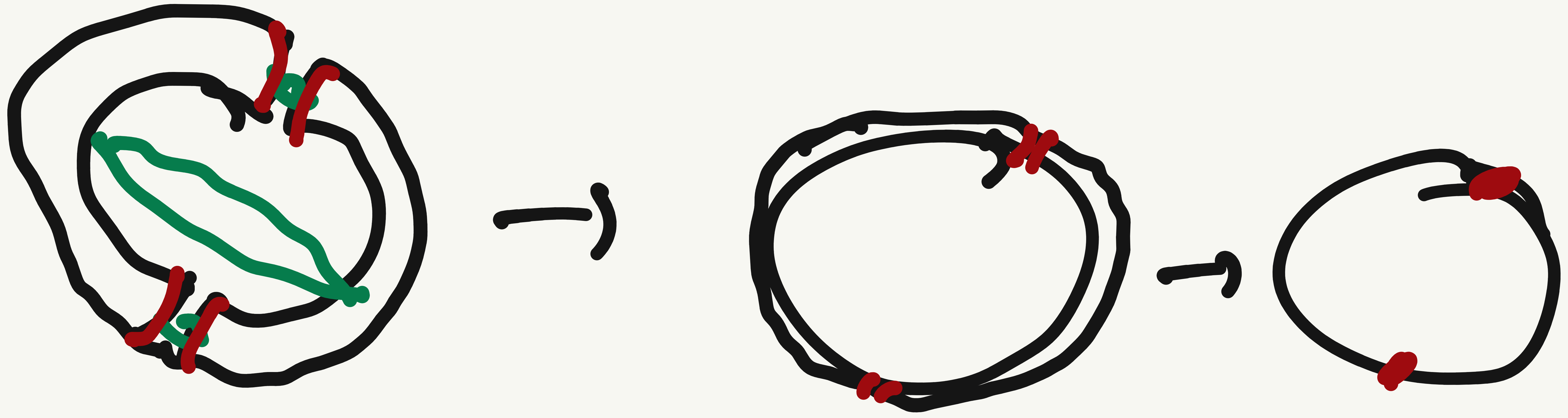
Rmk) $\sqrt{-2t}\,\Sigma$ is a MCF, and $H\nu - \frac{1}{2}x = 0$

Thm) Every rescaled flow $\tilde{M}_\tau$ either disappears or converges to a shrinker w/ multiplicity (in measure sense)

Ex) shrinkers



Potential Ex) Convergence w/ mult.



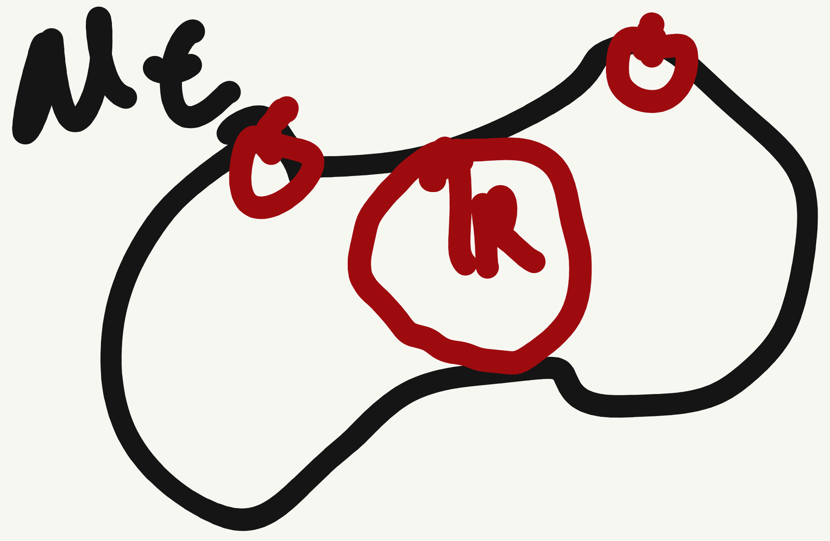
Thm) If M_0 is mean convex, then M_t remains mean convex.

pf) $H_t = \Delta_g H + |A|^2 H$, $\Delta_g = g^{ij}\partial_i\partial_j + g^{ij}\Gamma_{ij}^k \partial_k$.

By max prin, $H > 0$ $\forall t > 0$.

Def) α-noncollapsing.

Let $R(x,t)$ denote the radius of the largest round sphere tangent at $x \in M_t$ and enclosed by M_t.



If $R(x,t)\,H(x,t) \geq \alpha > 0$,

then we say that M_t is α-noncollapsed.

Thm) If M_0 is α- noncollapsed,
then M_t is " for $t>0$.

Ref) B. White 00', Sheng-Wang 09', B. Andrews 12'.

c.f. κ- noncollapsing of Ricci flow of Perelman. 02'.

proof) $Z : M^n \times M^n \times [0,T) \to \mathbb{R}$ (Andrews 12')

$$Z(p,q,t) = \alpha \langle X(p,t) - X(q,t), \nu(p,t) \rangle - \|X(p,t) - X(q,t)\|^2 H(p,t)$$

$Z \geq 0 \iff M_t$ is α- noncollapsed

If $Z(\cdot,\cdot,0) \geq 0$, then $Z(\cdot,\cdot,t) \geq 0$ $\forall t>0$ by the max. prin.

Cor) Blow-up limit of rescaled mean convex flow can NOT be a shrinker w/ multiplicity.

Thm) Huisken.

A colsed mean convex shrinker must be the round sphere

pf) $\Delta\left(\frac{|A|^2}{H^2}\right) + V\cdot\nabla\left(\frac{|A|^2}{H^2}\right) \geq \square^2 \geq 0$

Strong max prin $\Rightarrow$ $|A|^2 = H^2$ and so $\square^2 = 0$

$\Rightarrow \square^2 = 0 \Rightarrow \nabla A = 0.$ i.e. space form.

Cor) A convex shrinker must be a round sphere.

a round cylinder. or a hyperplane.

Thm) Huisken-Sinestrari. 99'

A blow-up limit of mean convex flow must be convex.

Namely, the limit shrinker must be a round sphere or a round cylinder.

pf) $\sigma_k = \sum \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_k}$. (ex. $\sigma_1 = H$. $\sigma_2 = R$. $\sigma_n = K$)

then. given $\eta > 0$ $\exists$ $C_{\eta, k}$ s.t.

$$S_k \geq -\eta H^k - C_{\eta, k}.$$

Thm) If M_0 is 2-convex. then M_t is 2-convex for $t > 0$

Moreover. $\frac{\kappa_1 + \kappa_2}{H} \geq \beta =: \min_{M_0} \frac{\kappa_1 + \kappa_2}{H} > 0$ holds for $t > 0$.

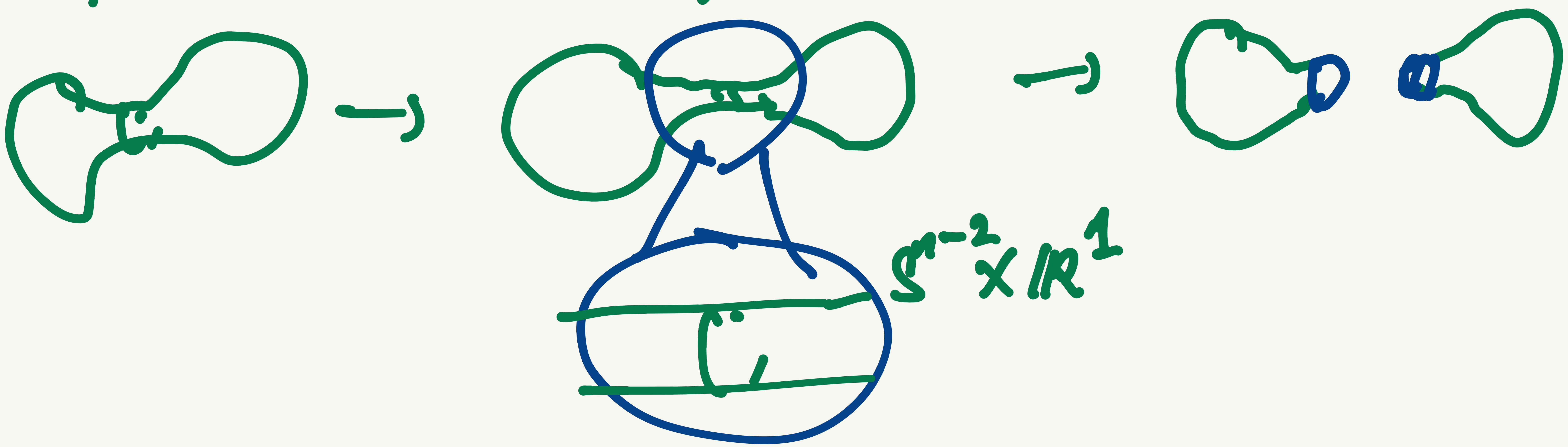
pf) $Z = k_1 + k_2$ is a concave operator. See Caffarelli–Cabre, "Fully nonlinear elliptic eq."

Hence, $Z_t \geq \Delta_g Z + |A|^2 Z \Rightarrow (Z-\beta H)_t = \Delta_g (Z-\beta H) + |A|^2 (Z-\beta H)$

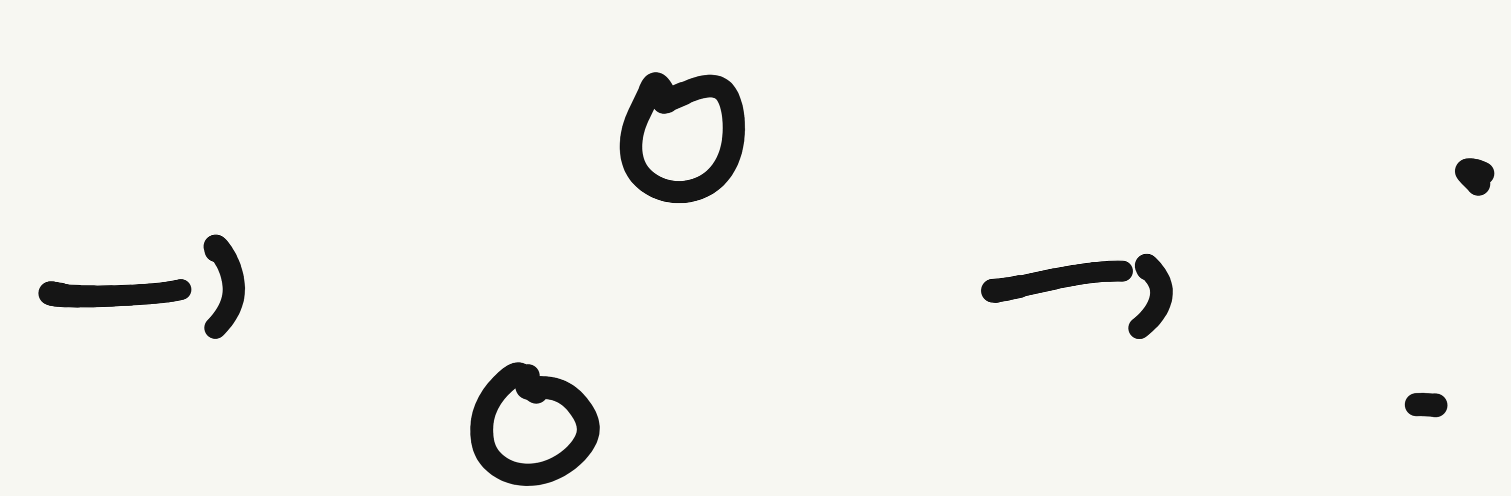
By max princ. $Z - \beta H \geq 0$. $\therefore k_1 + k_2 \geq \beta H$.

Cor) If M_0 is 2-convex, then a blow-up limit shrinker must be a round S^{n-1} or a round $S^{n-2} \times \mathbb{R}^1$.

Snapshot of Surgery)



$$\frac{\partial}{\partial x}\frac{xy}{x+y} = \frac{y}{x+y} - \frac{xy}{(x+y)^2} = \left(\frac{y}{x+y}\right)^2$$



By HS 09', BH 16', BHH 16'.

2-convex flow only needs finitely many surgery

$\Rightarrow$ 2-convex sphere is unknotted.