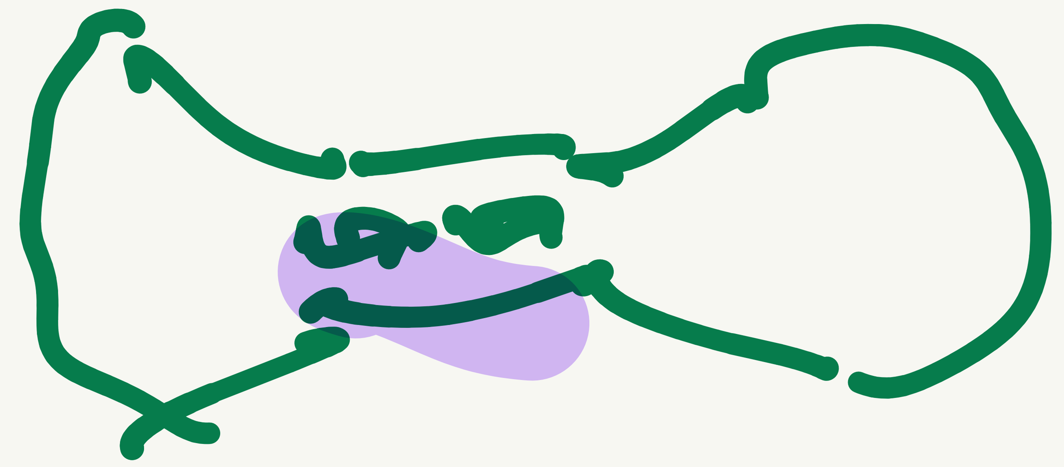
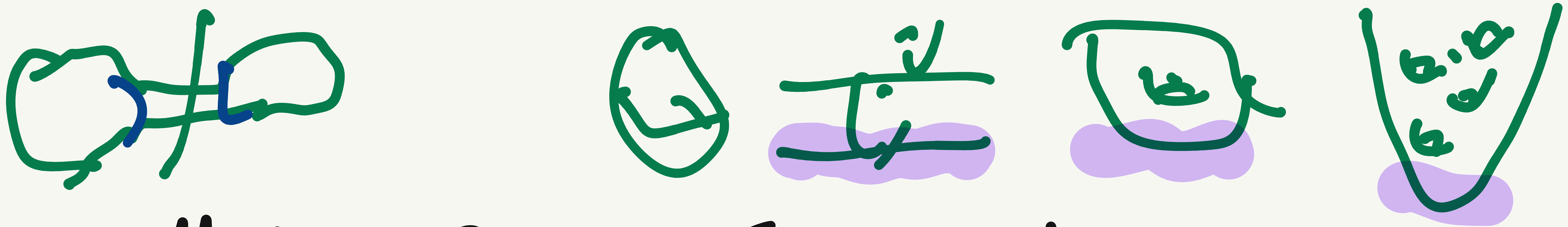
Day 4: Gradient flow and perturbation.



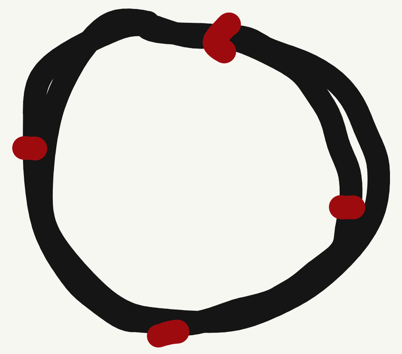
Recall



① $\sqrt{-2t}\,\Sigma \subset \mathbb{R}^{n+1}$ is a MCF $\iff$ Σ is a shrinker.

$\iff$ Σ satisfies $H = \frac{1}{2}\langle x, \nu \rangle$

② Every blow-up limit of a singularity must be a shrinker w/ multiplicity



③ Non-collapsed flow's blow-up limit must be a shrinker w/ multi 1.

④ Mean convex flow is non-collapsed.

⑤ Mean convex shrinkers are round $S^n, S^{n-1}\times\mathbb{R}^1, \cdots S^1\times\mathbb{R}^{n-1}$.

⑥ The surgery theory has been developed only for $S^{n-1}\times\mathbb{R}^1$.

Multiplicity one conjecture) A closed MCF has mult-1 blow-up limit at singularities.

Q1. What happens if we do not operate surgery...

Q2. Can we manipulate MCFs to have singularities w/ some good properties??

and... what are the good properties..??

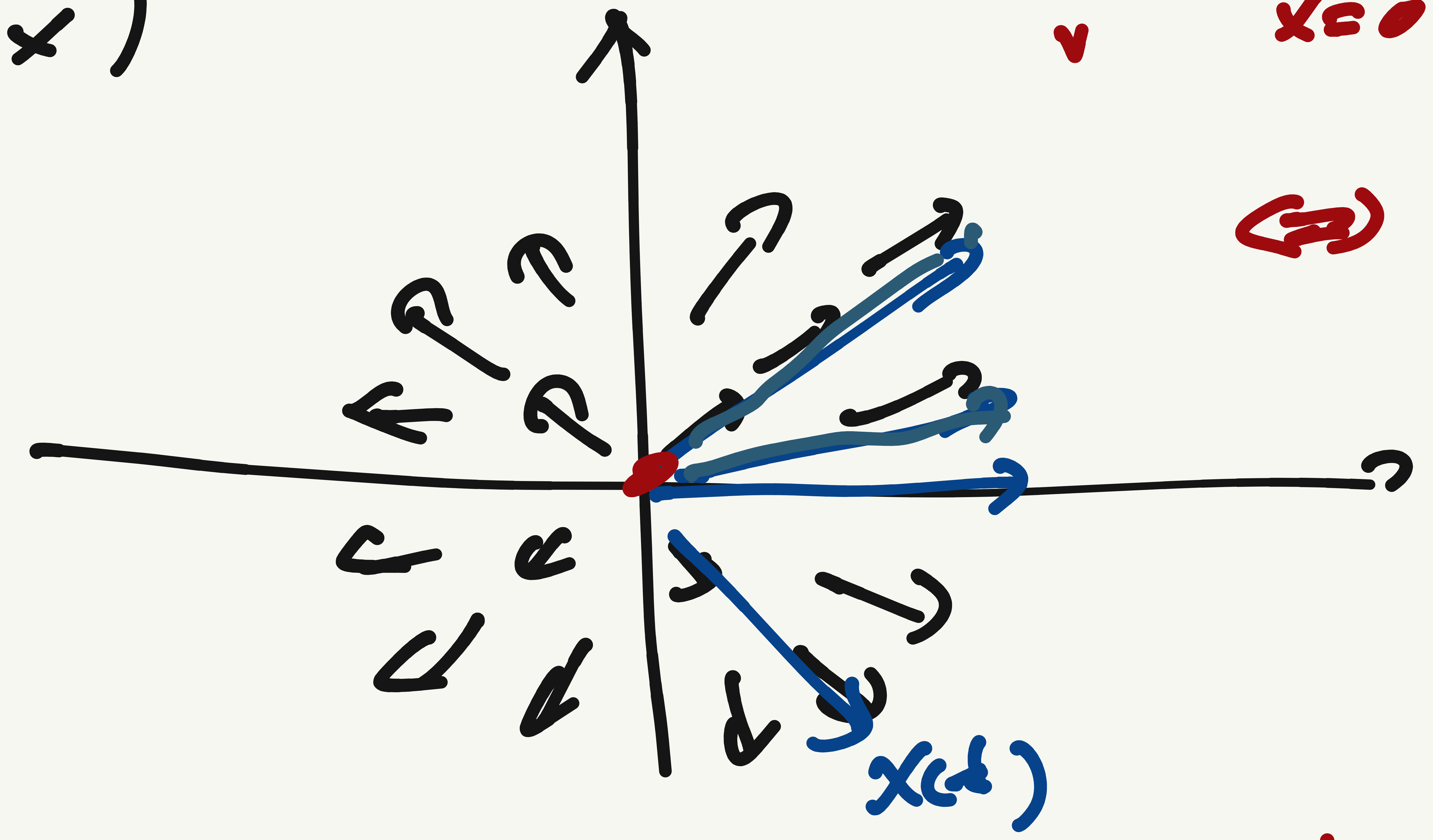
Gradient flow in $\mathbb{R}^n$.

Given $f: \mathbb{R}^n \to \mathbb{R}$ of class C^1, a curve $x: I \to \mathbb{R}^{n+1}$ satisfying

$$\frac{d}{dt}x(t) = -Df(x(t))$$

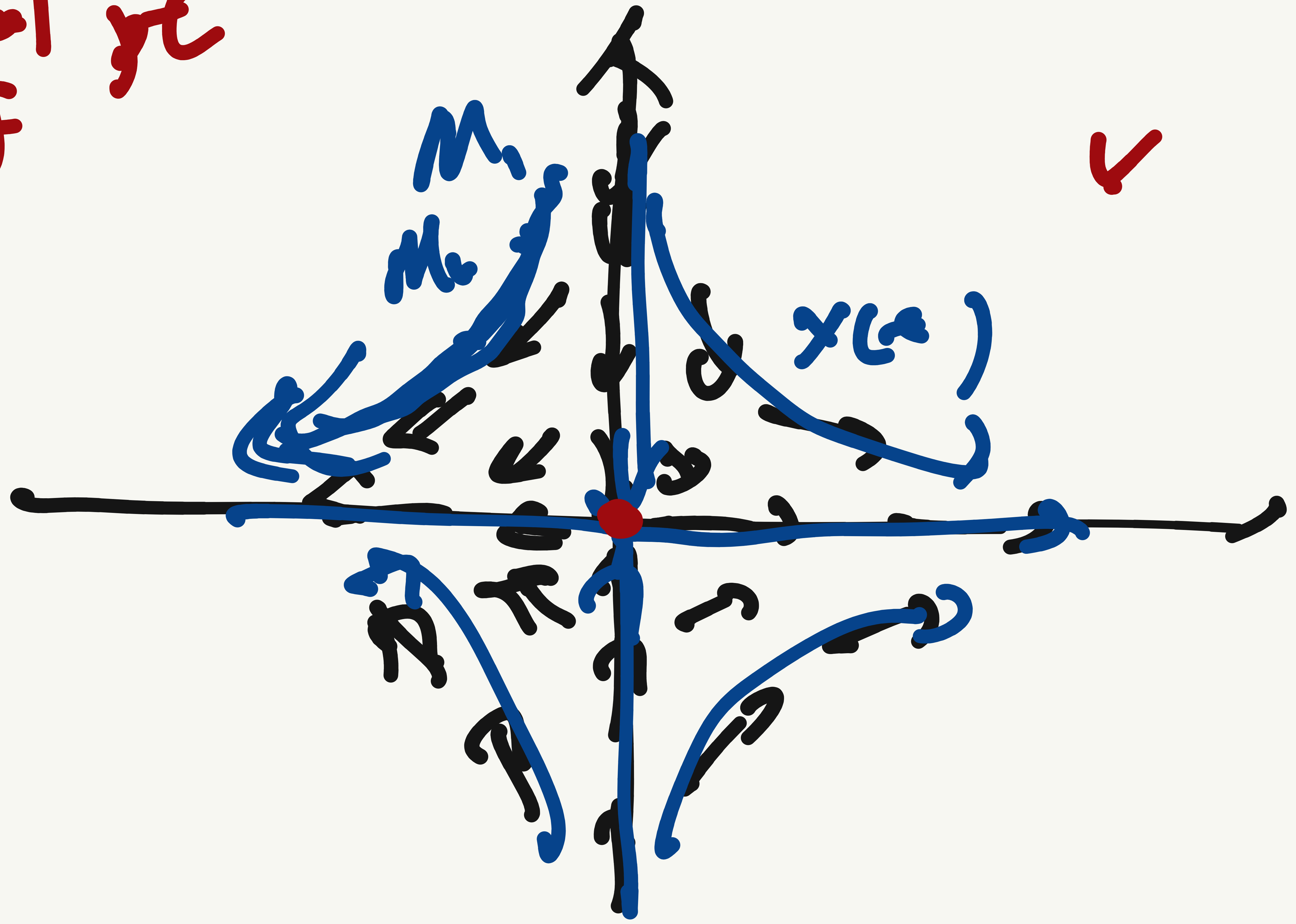
is a (negative) gradient flow of f.

Ex)



$x=0$ is critical pt of f

$\Leftrightarrow Df(x)=0$



$-f(x,y) = x^2+y^2$, $-Df = \begin{pmatrix}2x\\2y\end{pmatrix}$.

$-f(x,y) = x^2-y^2$, $-Df = \begin{pmatrix}2x\\-2y\end{pmatrix}$

MCF is a gradient flow of Entropy.

Colding-Minicozzi '12) We define the entropy of Σ by.

$$Ent(\Sigma) = \sup_{y \in \mathbb{R}^{n+1}, \lambda > 0} \int_\Sigma \frac{1}{(4\pi\lambda)^{\frac{n}{2}}} e^{-\frac{|x-y|^2}{4\lambda}} d g_{ay}.$$

Then by Huisken's monotonicity $\frac{d}{dt} Ent(M_t) \leq 0$. for MCF M_t.

Namely, every MCF is a gradient flow of Entropy $=: f$

$\mathbb{R}^n$ = space of surface

Remark) Every shrinker $\Leftarrow$ ~~is~~ a critical pt of Entropy.

Remark) Entropy is invariant under scaling, and rigid motion.

i.e. $Ent(\Sigma) = Ent(\lambda\Sigma) = Ent(\Sigma + y) = Ent(R\Sigma)$

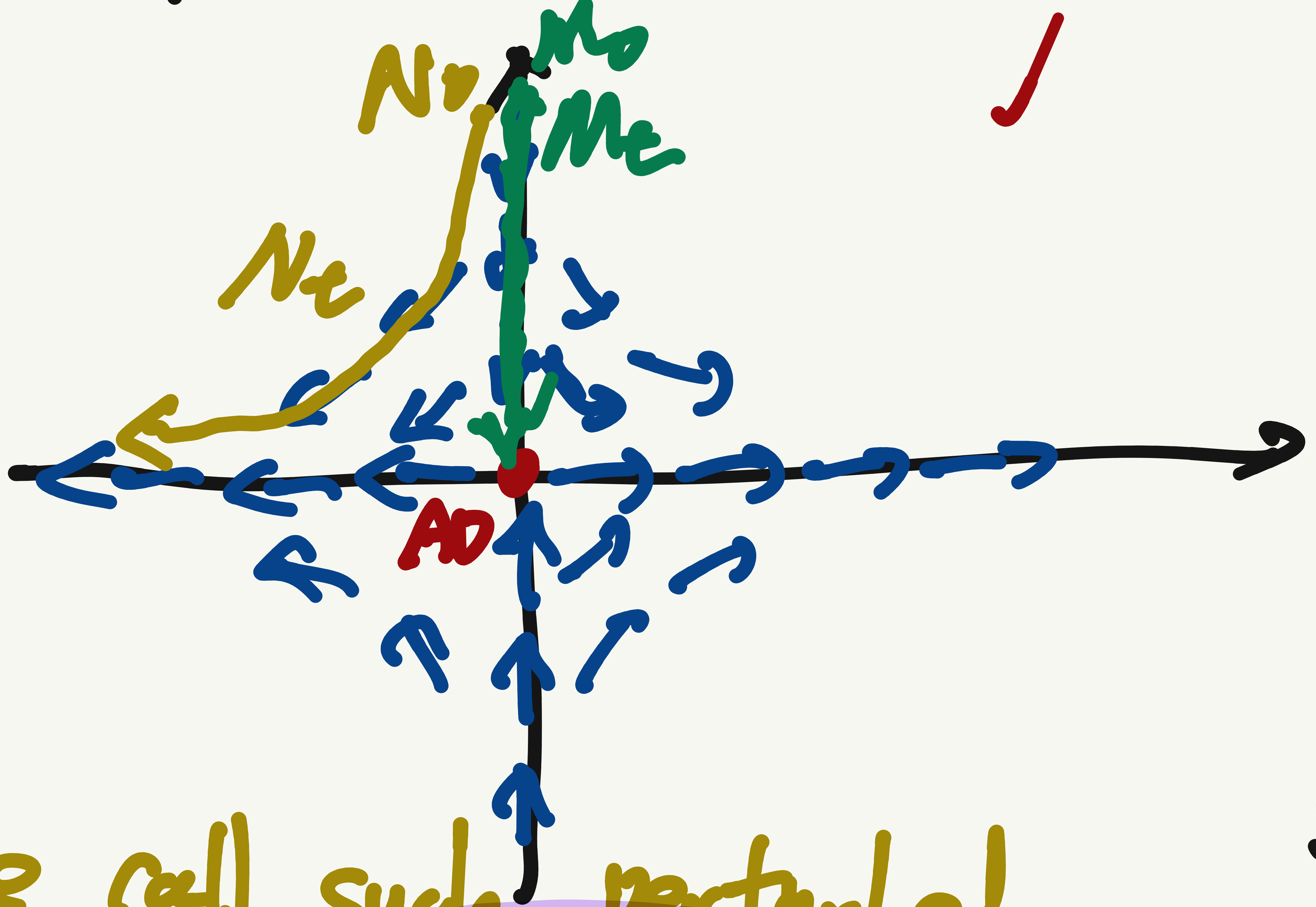
$\hookrightarrow$ rotation matrix.

We can resolve the difficulties in Q1 & Q2 by using gradient flow.

For example, we consider the case that M_t converges to the Angenent doughnuts 

after rescaling.

Then, the M_t looks like

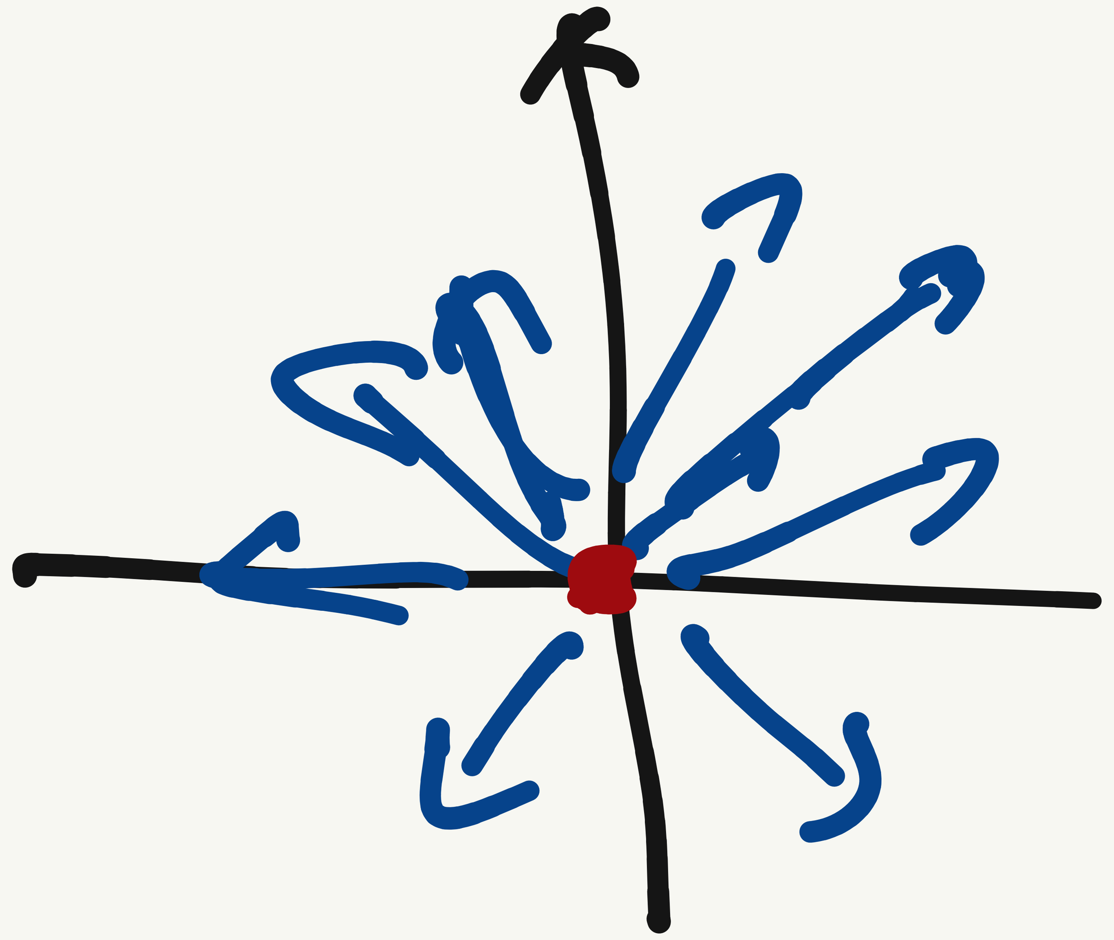


$\Rightarrow$ So, we can choose N_0 which is close enough to M_0, and the MCF N_t avoids AD-singularity of M_0.

Since N_0 and M_0 are ambient isotopy, we can consider N_t instead of M_t to study the Topo. of M_0.

we call such perturbed MCF N_t a generic MCF.

Stability of critical points

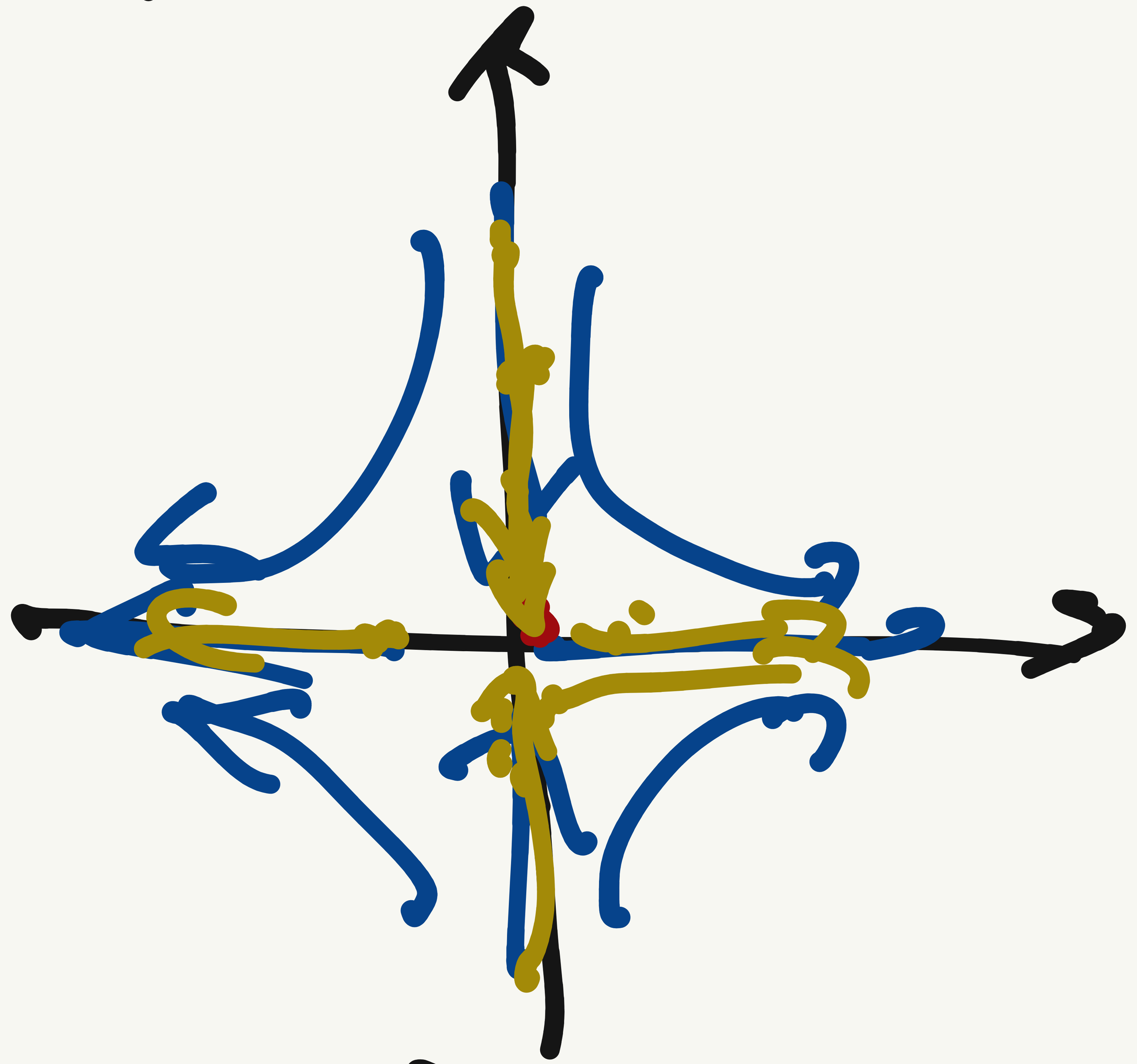


- $f = x^2+y^2$
- $D^2 f = \begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}$

positive eigenvalues

0 is unstable

Every vector is an eigenvector.



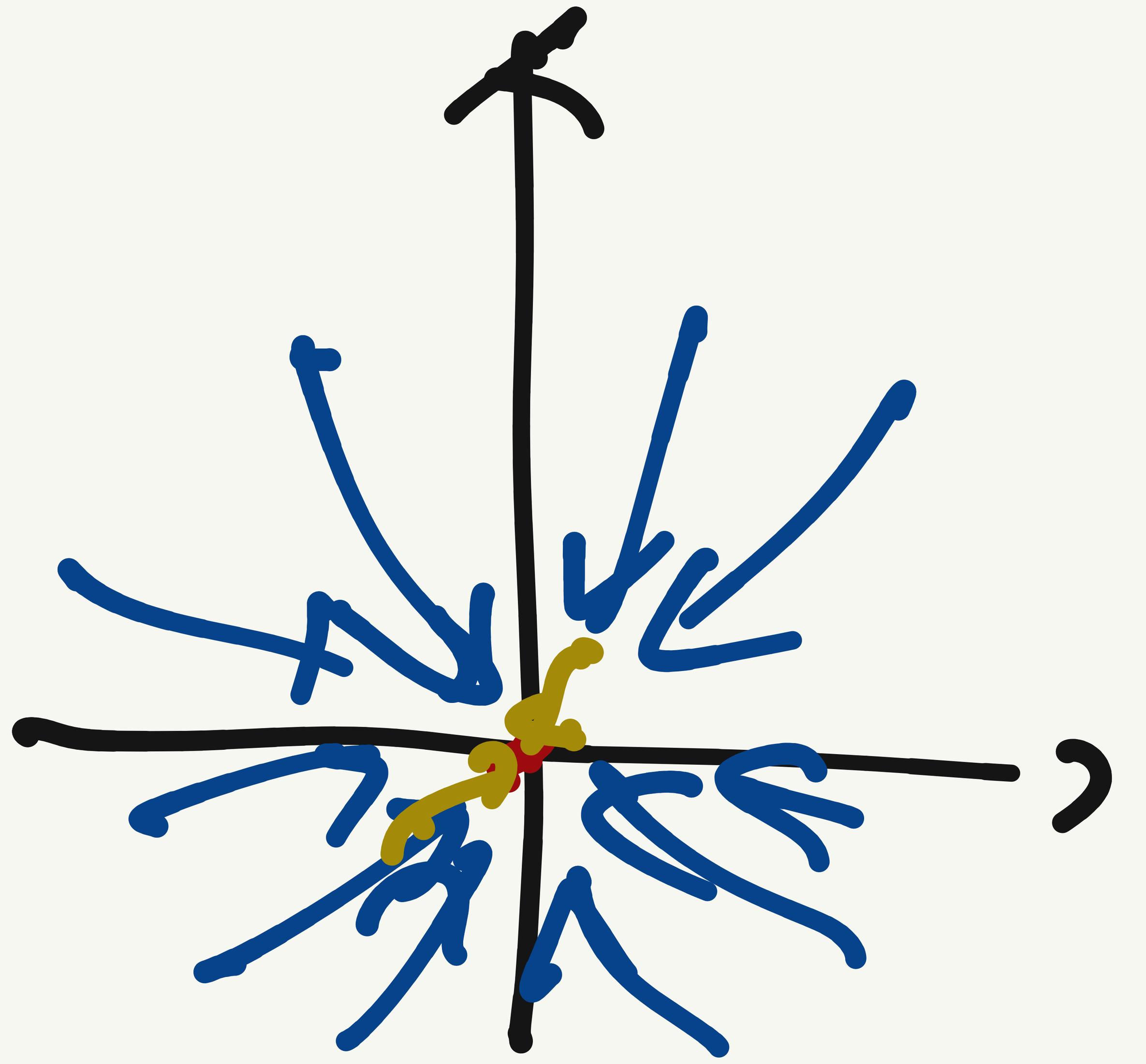
- $f = x^2-y^2$
- $D^2 f = \begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}$

mixed.

0 is unstable

(0,1), (1,0) are eigenvector

stable unstable

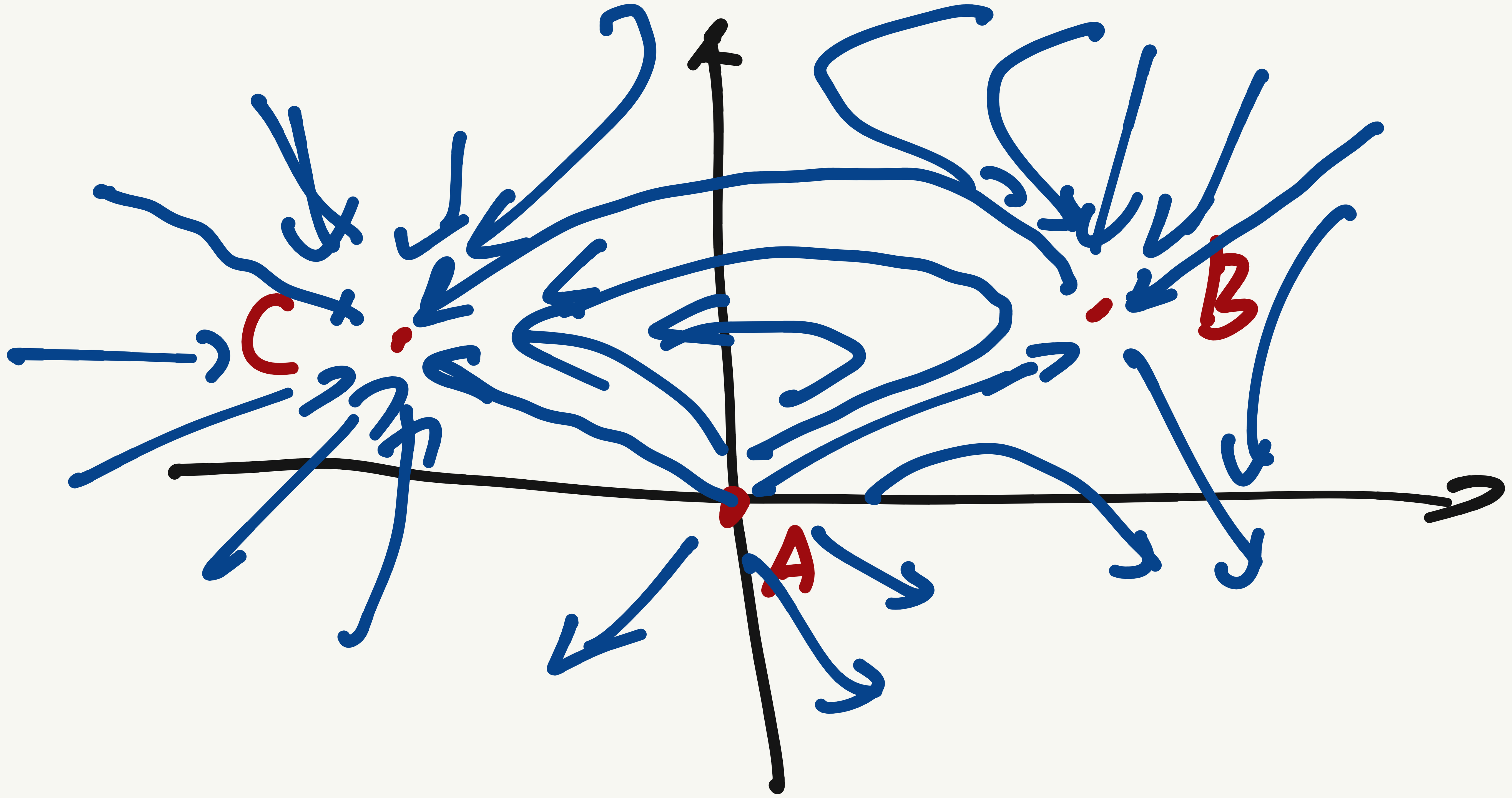


- $f = -x^2-y^2$
- $D^2 f = \begin{pmatrix}-2 & 0\\ 0 & -2\end{pmatrix}$

negative eigen.

0 is stable.

Every vector is eigenvector.



A, B are unstable
C is stable

Heat eq is a gradient flow.

$u: \mathbb{R} \times [0,+\infty) \to \mathbb{R}$, $u(x+2\pi,t) = u(x,t)$, $u_t = u''$.

Consider $E(u) = \int_0^{2\pi} |u'|^2 dx$.

$$\Rightarrow \frac{d}{dt} E(u) = 2\int_0^{2\pi} u' u'_t \, dx \underset{\text{integration by parts}}{=} 2\int_0^{2\pi} -u'' u_t \, dx = -2\int_0^{2\pi} |u_t|^2 dx \leq 0.$$

Eigenfunction vs Eigenvector.

$\varphi'' + \lambda\varphi = 0$, $\varphi(x+2\pi) = \varphi(x)$ $\Rightarrow$ φ is an eigenfunction w/ eigenvalue λ.

$(A\varphi = -\lambda\varphi)$

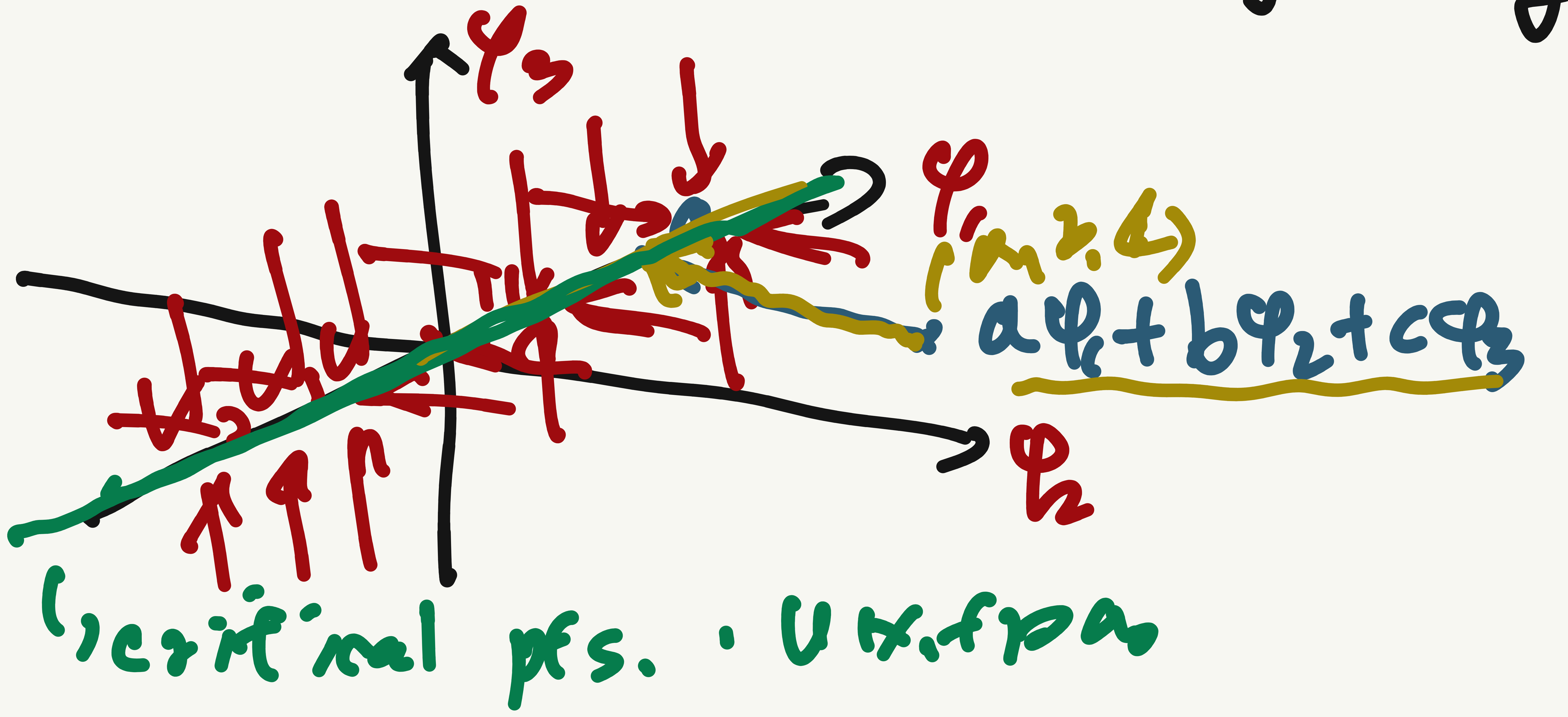
$\Rightarrow$ Eigenfunctions & Eigenvalues.

$1'' + 0 \cdot 1 = 0 + 0 = 0$ $\quad$ $(\cos\theta)'' + 1 \cdot \cos\theta = -\cos\theta + \cos\theta = 0$

$(1, 0)$, $(\sin\theta, 1)$, $(\cos\theta, 1)$, $(\sin 2\theta, 4)$, $(\cos 2\theta, 4)$, $\cdots$

$\varphi_1 \quad \varphi_2 \quad \varphi_3$

and linear combination of eigenfunctions of the same eigenvalue.



$$\boxed{u(x,t) = a_0 + \sum_{m=1}^{\infty} a_m e^{-m^2 t} \cos m\theta + \sum_{m=1}^{\infty} b_m e^{-m^2 t} \sin m\theta}$$

pf) Let $\langle f,g\rangle = \int_0^{2\pi} fg\,dx$, $f,g \in L^2([0,2\pi])$

$\Rightarrow \left\{ \frac{1}{2\pi}, \frac{1}{\pi}\cos\theta, \frac{1}{\pi}\sin\theta, \frac{1}{\pi}\cos 2\theta, \cdots \right\}$ is an orthonormal basis of the L^2-space.
$\quad\ \ \varphi_1 \qquad \varphi_2 \qquad \varphi_3 \qquad \cdots$

$\Rightarrow \varphi_{2m}'' + m^2\varphi_{2m} = 0$. $\varphi_{2m+1}'' + m^2\varphi_{2m+1} = 0$ ✓

and $u(x,t) = \sum_{m=1}^{\infty} \langle u(x,t), \varphi_m\rangle \varphi_m$

$\frac{d}{dt}\langle u(x,t), \varphi_{2m}(x)\rangle = \int_0^{2\pi} u_t\varphi_{2m} = \int_0^{2\pi} u''\varphi_{2m} = \int_0^{2\pi} -u'\varphi_{2m}'$

$= \int_0^{2\pi} u\varphi_{2m}'' = -m^2\int_0^{2\pi} u\varphi_{2m} = -m^2\langle u(x,t), \varphi_{2m}(x)\rangle$

$\Rightarrow \langle u(x,t), \varphi_{2m}(x)\rangle = e^{-m^2t}\langle u(x,0), \varphi_{2m}(x)\rangle$ $:a_{2m}$ $\square$.

Thm) $\Omega \subset \mathbb{R}^n$ be a bound open set.

$$u_t = \Delta u + V(x)\, u(x,t) \quad \text{holds in } \Omega \times [0,T) \longleftarrow (\ast)\ 1$$

$$u = 0 \quad \text{" on } \partial\Omega \times [0,T)$$

$$\Rightarrow \quad u = \sum_{m=1}^{\infty} a_m e^{-\lambda_m t} \varphi_m(x).$$

where $\{\varphi_1, \cdots \}$ is an orthonormal basis of $L^2(\Omega)$

$$\varphi_m'' + \lambda_m \varphi_m = 0. \qquad \varphi_m = 0.$$

Thm) $\varphi_m > 0 \iff m = 1.$

Ex) $u_t = u'' + 9u$, $\quad u(0,t) = u(2\pi,t) = 0$

$$E(u) = \int_0^{2\pi} u'^2 - 9u^2$$

$$\Rightarrow \frac{d}{dt}E = 2\int u'u_t' - 9uu_t = -2\int u_t(u''+9u)$$

$$= -2\int u_t^2 \le 0.$$

Eigenpairs $(1,-9)$, $(\cos\theta, -8)$, $(\sin\theta, -8)$ unstable

$(\cos 2\theta, -4)$, $(\sin 2\theta, -4)$

$(\cos 3\theta, 0)$, $(\sin 3\theta, 0)$ neutral

$(\cos 4\theta, 7)$, $(\sin 4\theta, 7)$, $\cdots$ stable

$u_0 = \cos\theta \Rightarrow u(\theta,t) = e^{8t}\cos\theta \to \infty$ as $t \to +\infty$.

Very important remark)

Suppose $|u| \le C$ for $t \in [-\infty, 0]$, (ancient solution of (*))

Then. $u(x,t) = \sum_{m=1}^{I+K} a_m e^{-\lambda_m t} \varphi_m(x)$. ← finite parameter family of ancient sol

where $\lambda_1 < \lambda_2 \le \cdots \le \lambda_I < 0$

$0 = \lambda_{I+1} = \cdots = \lambda_{I+K}$,

$0 < \lambda_{I+K+1} \le \lambda_{I+K+2} \le \cdots$

I ; Morse index

K ; nullity.