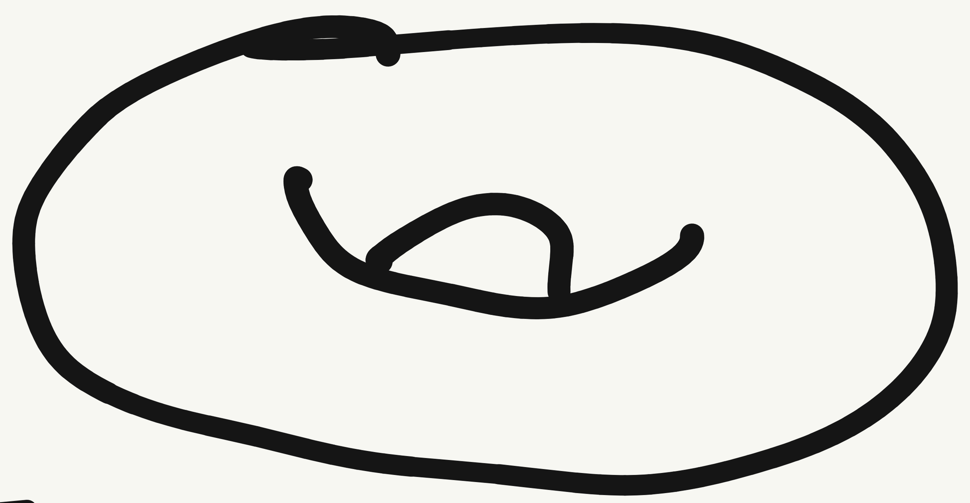
# Ancient mean curvature flow



($\sqrt{-t}\,\Sigma$ is MCF)

Given an ancient MCF $M_t$ ($t \in (-\infty, T)$)
we consider the rescaled ancient flow $\tilde{M}_\tau$

$$\tilde{M}_\tau = \frac{1}{\sqrt{-t}} M_t \quad , \quad \tau = -\log(-t)$$

Suppose that $\tilde{M}_\tau$ converges to $\Sigma$ in $C^k_{loc}$ $(k \geq 2)$ as $\tau \to -\infty$.

Namely. $\exists\, u : \Sigma \times (-\infty, \tau_0) \to \mathbb{R}$ s.t.

$$x + u(x,\tau)\,\nu(x) \in \tilde{M}_\tau \quad \forall x \in \Sigma,\ \tau < \tau_0,$$

$$\|u(\cdot,\tau)\|_{C^k(\Sigma)} \to 0 \quad \text{as} \quad \tau \to -\infty.$$

Then, $u_\tau = Lu + E(u)$, where $|E(u)| \leq C\|u\|_{C^1}\|u\|_{C^2}$.

$$Lu = \Delta_\Sigma u - \frac{1}{2} x^{tan} \cdot \nabla_\Sigma u + \left(|A_\Sigma|^2 + \frac{1}{2}\right)u.$$

Let $\langle f,g\rangle_w =: \int_\Sigma fg\, e^{-\frac{|x|^2}{4}}$. Then. $\langle Lf,g\rangle_w = \langle f, Lg\rangle_w$

$\Rightarrow \exists\ (\varphi_1,\lambda_1), \cdots$ s.t. $L\varphi_i + \lambda_i \varphi = 0$, $\lambda_i \to +\infty$ as $i \to +\infty$

$\mathrm{span}\{\varphi_1, \varphi_2, \cdots\} = L^2_w(\Sigma) = \{f : \Sigma \to \mathbb{R} \mid \int_\Sigma f^2 e^{-\frac{|x|^2}{4}} < +\infty\}$

$\lambda_1 < \lambda_2 \le \cdots \le \lambda_I < 0 = \lambda_{I+1} = \cdots = \lambda_{I+K} = 0 < \lambda_{I+K+1} \le \cdots$,

$\varphi_m > 0 \iff m = 1$.

Recall that an ancient sol $V_\tau = LV$ w/ $|V| \le c$ must be

$$V = \sum_{i=1}^{I} a_i e^{-\lambda_i \tau} \varphi_i(x) + \sum_{i=I+1}^{I+K} a_i \varphi_i(x). \quad - (*)$$

Since $u_\tau = Lu + E(u) \approx V_\tau = LV$, $u$ would behave like $(*)$

$(\because |E(u)| \le c\, \|u\|_{C^1} \|u\|_{C^2})$

[CM19] ① If $|u| \le e^{\delta\tau}$, then $\lim_{\tau\to-\infty} u e^{\lambda_i \tau} = \varphi$ for some $i \le m$ and $L\varphi + \lambda_i \varphi = 0$.

①' If $u^1, u^2$ are in the case $\lambda_i = \lambda_1$, $u^1(\cdot,\tau) = u^2(\cdot, \tau + A)$ for some $A \in \mathbb{R}$

② $\exists$ continuous map $S : B_\varepsilon(0) \subset \mathbb{R}^I \to C^\infty(\Sigma \times (-\infty, 0])$ s.t. $u_{\vec{a}} = S(\vec{a})$ solves $\partial_\tau u_{\vec{a}} = L u_{\vec{a}} + E(u_{\vec{a}})$

③ If an ancient solution $u$ does not satisfies $|u| \le e^{\delta\tau}$ $\forall \delta > 0$, then $\lim_{\tau\to-\infty} \frac{u}{\|u\|_w} = \varphi$ for some $L\varphi = 0$.

ADS 19, ADS 20) The case that such $\varphi$ in ③ exists.

CS20a, CS20b) 〃 〃 does NOT exist.

Remark) If $\lambda_i = \lambda_1$, then $u \approx a e^{\lambda_1 \tau} \varphi_1$
$\Rightarrow u, u_\tau > 0$ or $u, u_\tau < 0$.

Cor) If $u > 0$ $\forall \tau < T_0$,

then $\lim_{\tau \to -\infty} e^{\lambda_1 \tau} u = a \varphi_1$ $(***)$ for some $a > 0$.

Namely, if $M_\tau$ stays in the one-side of $\Sigma$,

then $(***)$ holds.

Fact) $LH + H = 0$, $Lx^i + \frac{1}{2} x^i = 0$, where $x^i = \langle x, e^i \rangle$

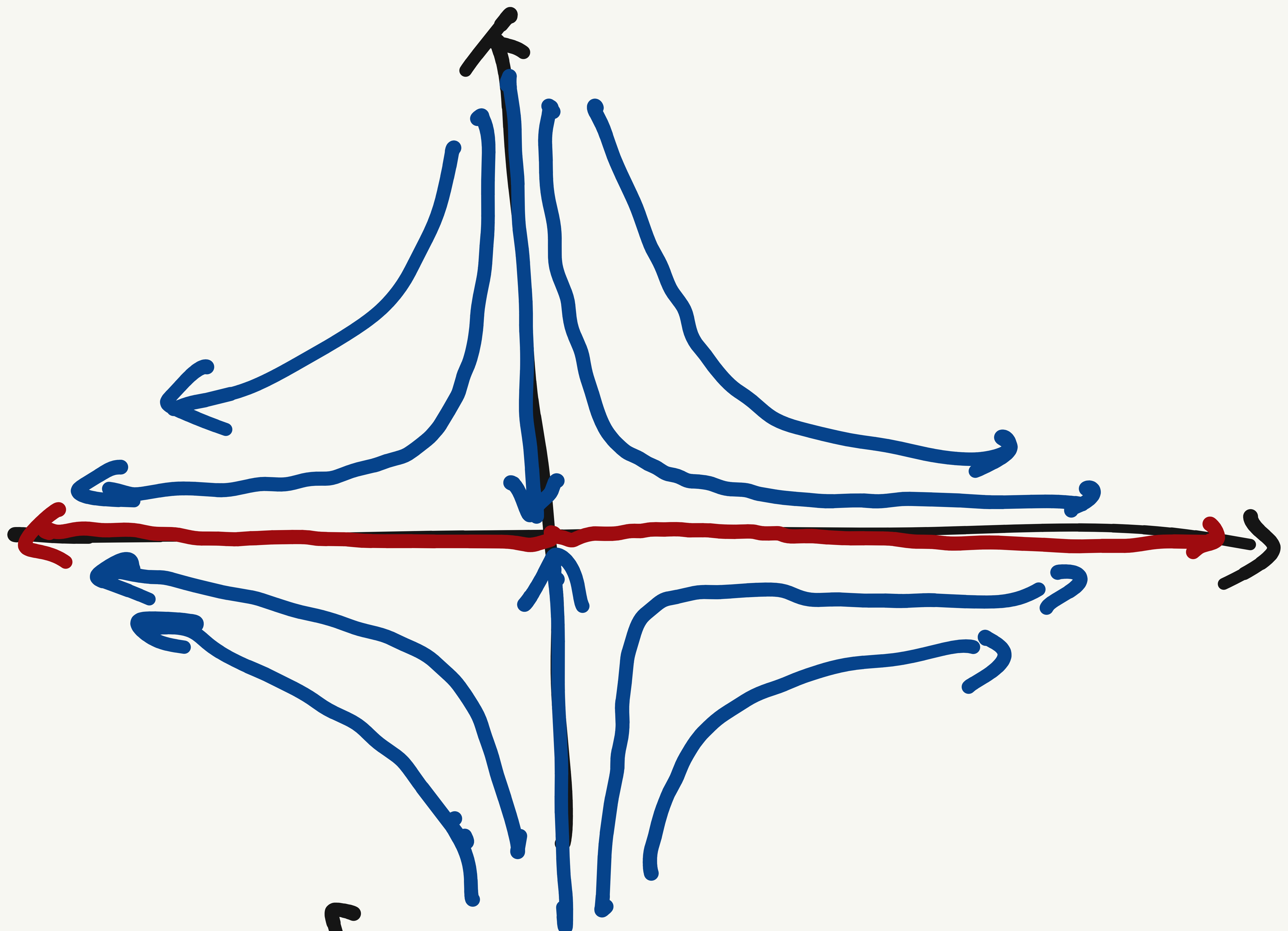
$\Rightarrow \exists (n+2)$-unstable eigenfunction.

What is the geometric meaning?

$M_t = a e^{-\frac{\tau}{2}} \Sigma$ w/ $1 \neq a > 0$ $\Rightarrow$ $u e^{-\tau} \to b H_\Sigma$

$M_t = e^{-\frac{\tau}{2}} (\Sigma + y)$ w/ $y \neq 0$ $\Rightarrow$ $u e^{-\frac{\tau}{2}} \to a_1 x^1 + \cdots + a_n x^n$.

i.e. Ancient flows along $H$ or $x^i$ are shrinking flow.



$-f = x^2 - y^2$

Ancient (negative) gradient flow

$$x(t) = (ae^{2t}, 0)$$

$$\Rightarrow \dot{x} = \begin{pmatrix} 2ae^{2t} \\ 0 \end{pmatrix} = -Df(x(t))$$

Hence, ancient flows escape from the critical point.

$\Rightarrow$ Ancient rescaled MCFs escape from shrinkers

+ Ancient RMCFs along $\underbrace{H, x^i}_{\text{trivial unstable eigenfunc}}$ has the same shape to shrinkers.

$\Rightarrow$ We can expect that a shrinker $\Sigma$ w/ non-trivial unstable eigenfunctions is unstable.

Def) Suppose that $H, x^i$ are the only unstable eigenfunctions of $L$ over $\Sigma$. Then, we say $\Sigma$ is linearly stable.
otherwise, $\Sigma$ is linearly unstable

Classification of linearly stable 4D Ricci shrinkers remains open.

CM 12) Round $S^n$, $S^{n-1}\times\mathbb{R}^1$, $\cdots$, $S^1\times\mathbb{R}^{n-1}$, and hyperplane $\mathbb{R}^n$ are the only linearly stable shrinkers

c.f. HS 99)

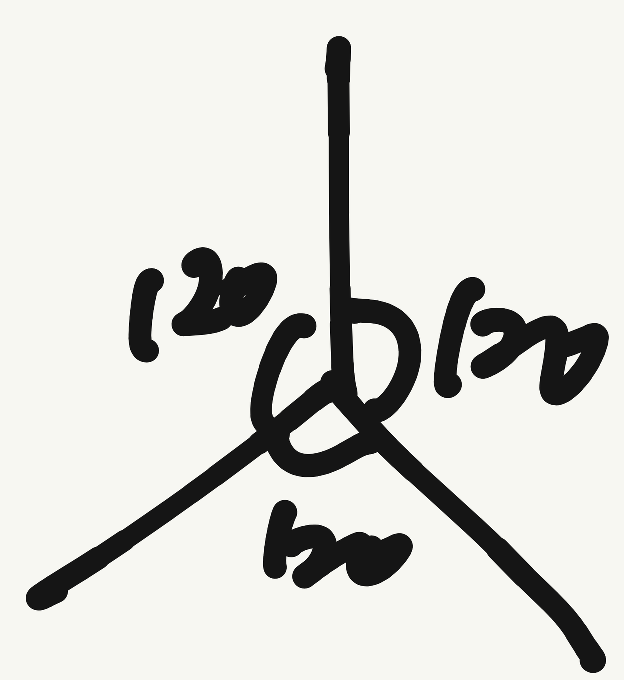
Remark) The first eigenvalue of linearly stable $\Sigma$ is $H$.
Thus, $H>0 \Leftrightarrow \Sigma$ is linearly stable

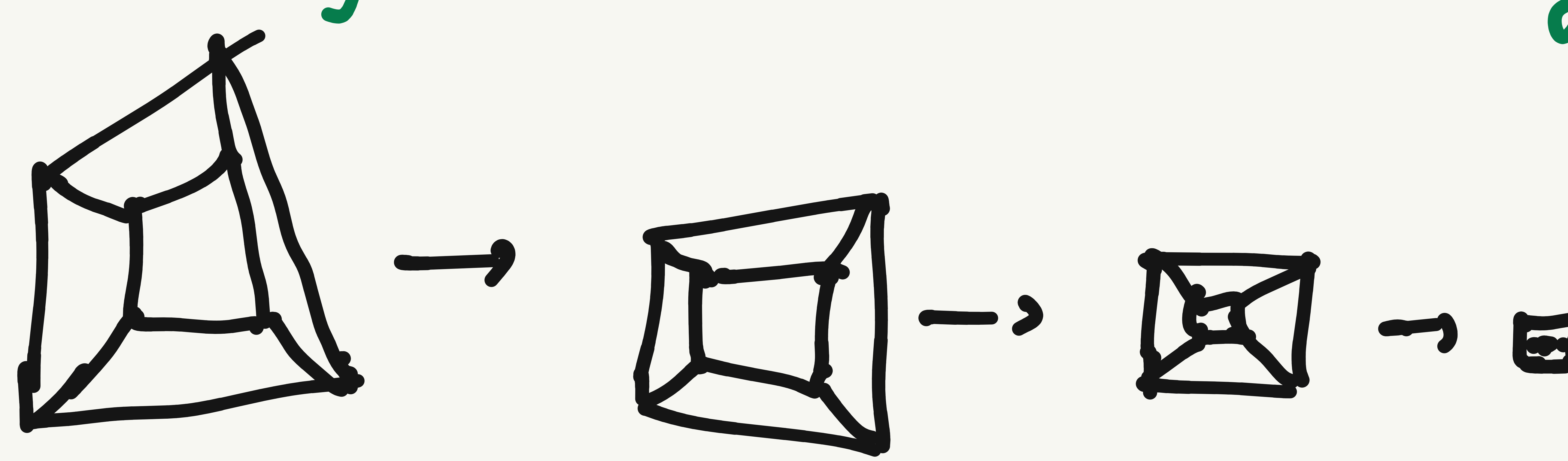
Huisken conjecture) Given a closed smooth embedded hypersurface $M_0 \subseteq \mathbb{R}^{n+1}$, there exist $\varepsilon$-graph $N_0$ over $M_0$ such that the Brakke flow $N_t$ only has multiplicity one round sphere or round cylinder singularities.

c.f. Ch 7. Gilbarg–Trudinger.

The Brakke flow is a $\boxed{\text{weak}}$ mean curvature flow which is NOT necessarily $C^\infty$ and can have singularities.

Ex)



1D Brakke flow.

Q) Does the Brakke flow is well-posed (well-defined) through the multi-1 cylindrical singularities ?

Mean convex neighborhood conjecture)

Suppose that at $(x_0, t_0) \in M_t$, a Brakke flow has multi-1 round spherical or round cylindrical singularity. Then, $\exists\ \varepsilon > 0$ s.t. $M_t$ has the positive mean curvature in $B_\varepsilon(x_0) \times (t_0 - \varepsilon, t_0 + \varepsilon)$

HW20) The conjecture implies the well-posedness in the nbh. And if every limit flow at $(x_0, t_0)$ is convex, then the conjecture is true.

Def) Limit flow at $(x_0, t_0)$

Let $(x_i, t_i, \lambda_i) \to (x_0, t_0, +\infty)$

Then. any subsequential limit

$$\bar{M}_t = \lim_{i \to +\infty} \lambda_i \left( M_{\lambda_i^{-2}(t-t_i)} - x_i \right)$$

is a limit flow at $(x_0, t_0)$ of $M_t$.

Remark) limit flow is an ancient flow.

Def) If $(x_i, t_i) = (x_0, t_0)$, then we call the limit flow

a tangent flow

(c.f. tangent cone)

Thm) Every tangent flow $\bar{M}_t$ w/ RMCF $\bar{M}_\tau$

converges to a shrinker w/ multiplicity

All the tangent flows at $(x_0, t_0) \in M_t$ are the same up to rotation.

Remark) Uniqueness of tangent flow

$$\iff \lim_{t \to t_0} \sqrt{t_0 - t}\,(M_t - x_0) = \Sigma \text{ w/ multi-m.}$$

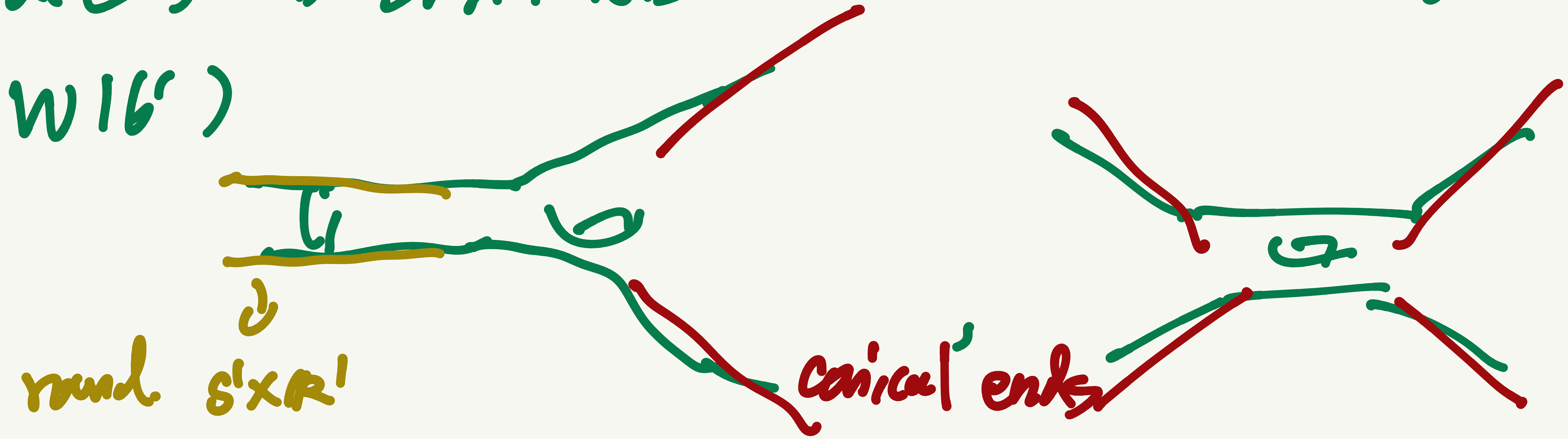
Multi-one case) S 14': closed shrinkers

CM 15': round cylinders

CS 19': Asymptotically conical shrinkers.

Fact) 2D shrinkers in $\mathbb{R}^3$ is smooth

WLOG)




Every end of 2D shrinker must be asymptotically round cylinder or cone. w/ multi-1.

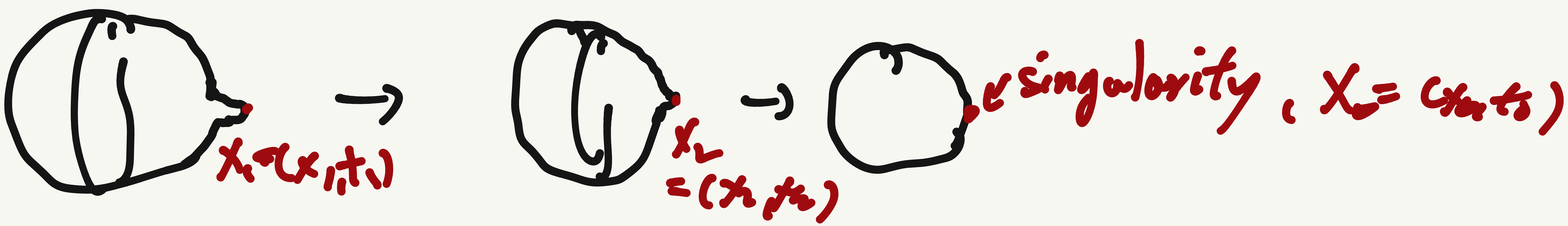
No cylinder conjecture) If a 2D shrinker has at least one cylindrical end, the it must be the round $S^1 \times \mathbb{R}^1$.

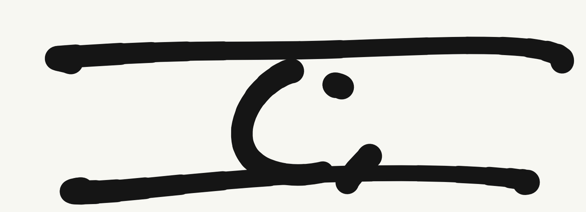
c.f. W 14', W 16'

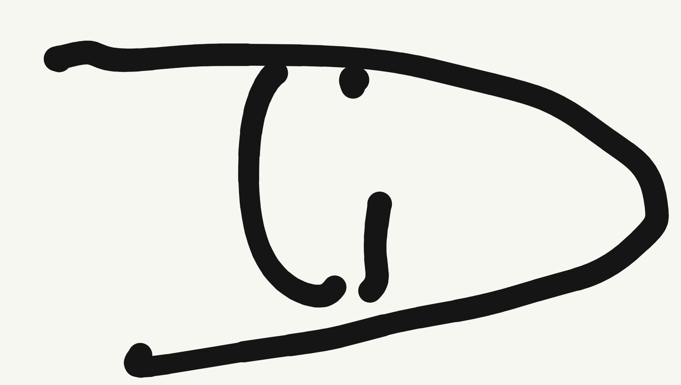
Remark) So, we expect that a 2D shrinker must be closed, round cylinder, or asymptotically conical.

Limit flow $\neq$ tangent flow

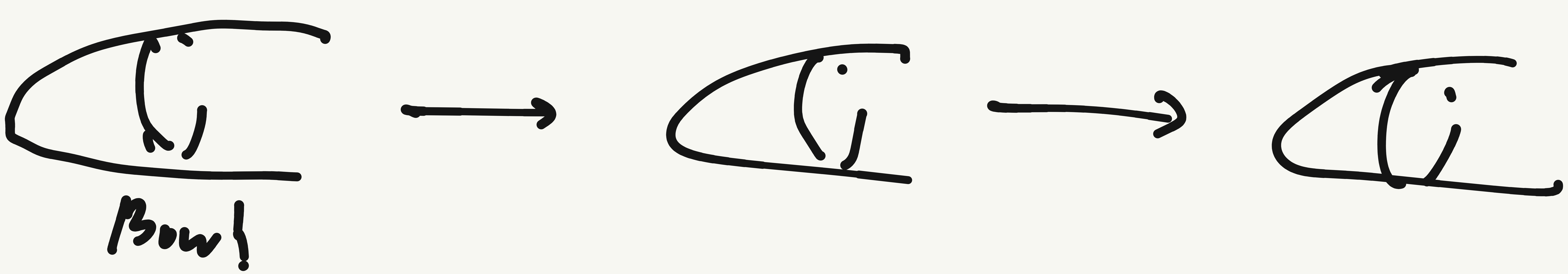
Degenerated neck ) Aw 94!




Tangent flow at $X_0$ is  round cylinder

Limit flow along $X_i$ is  a bowl soliton

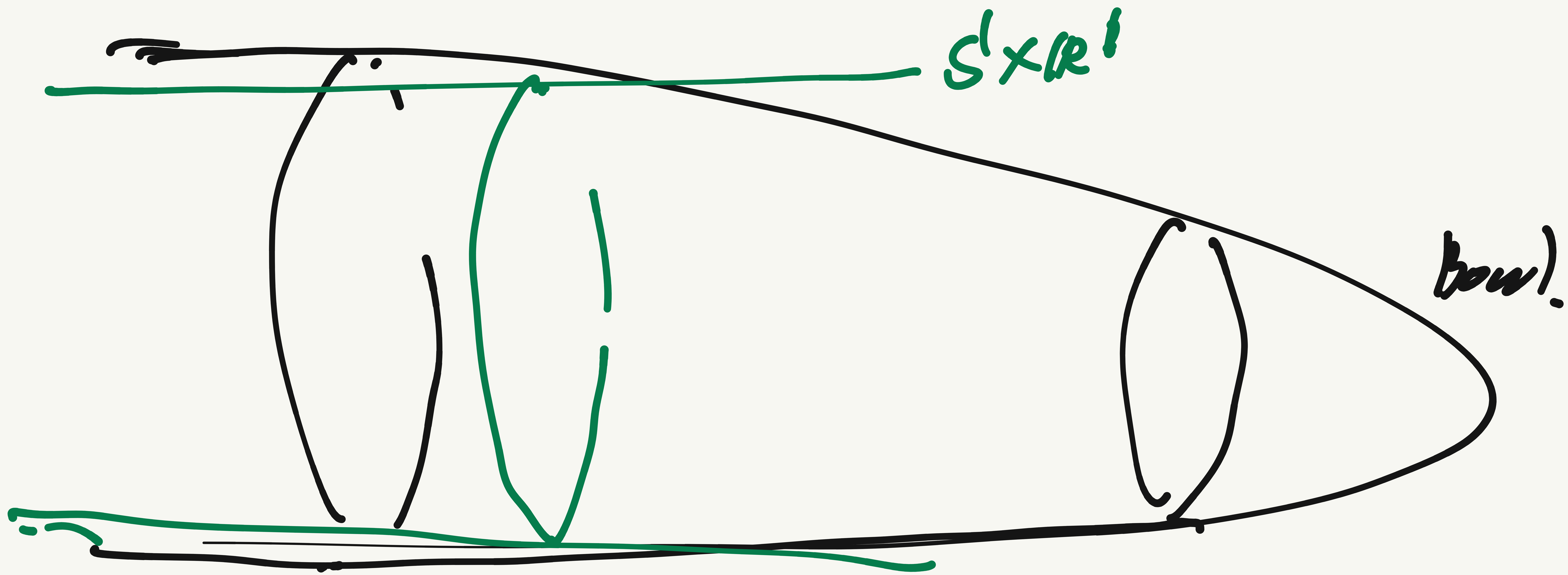
Bowl soliton is a TRANSLATING flow w/ rot. sym.



Remark) Let $M_t$ be a bowl soliton

$\tilde{M}_\tau = (-t)^{-1/2} M_t$, $\tau = -\log(-t)$ is RMCF.

$\Rightarrow$ $\tilde{M}_\tau \to$ round cylinder as $\tau \to -\infty$

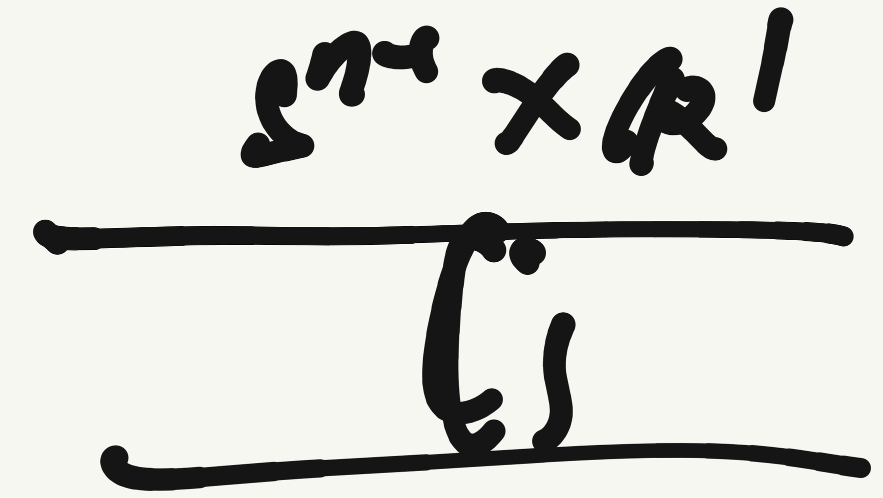


(CHH18') The mean convex neighborhood conjecture is true in $\mathbb{R}^3$

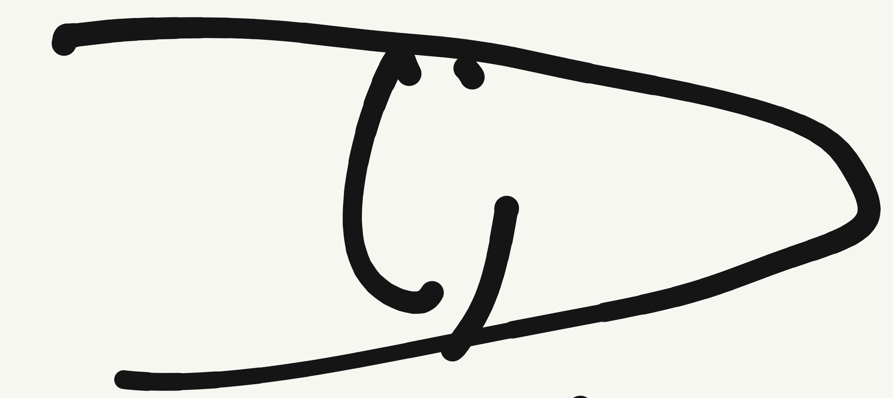
(CHHW'19') The conjecture is true at $S^n$, $S^{n-1}\times\mathbb{R}^1$ singularities in $\mathbb{R}^{n+1}$.

Snapshot of proof) A ancient flow from $S^{n-1}\times\mathbb{R}^1$ must be


round sphere,

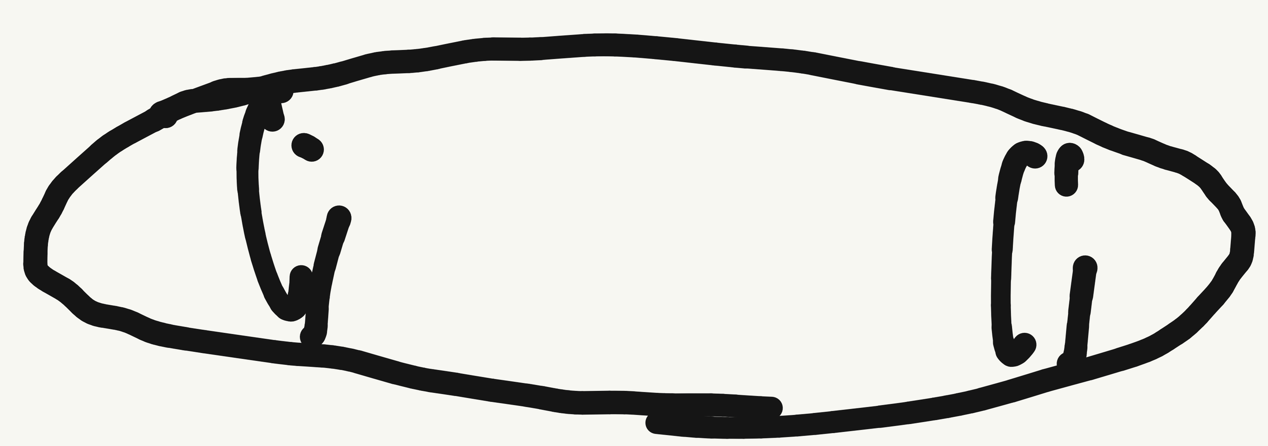



round cylinder,



Bowl.

or



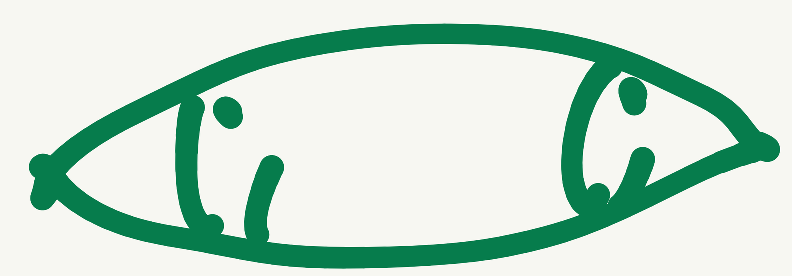
Ancient oval.

$\Rightarrow$ convex & rotationally symmetric

CM 15') uniqueness of tangent flow at cylindrical sing.

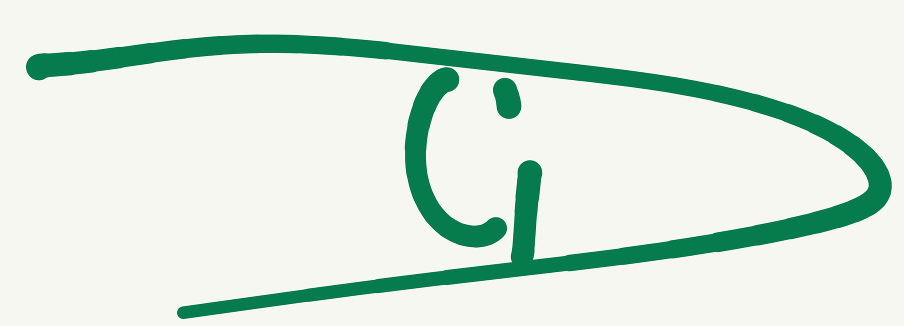
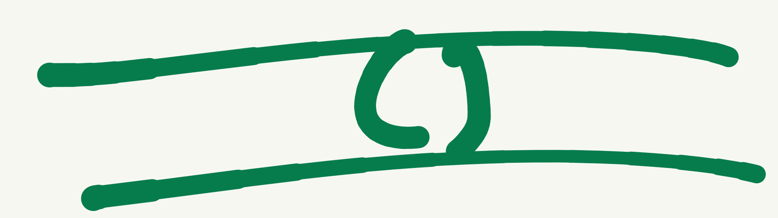
W 05') Local regularity theory.

ADS 19', 20') classification of convex, uniformly 2-convex, non-collapsed ancient flow

$\Rightarrow$  

sphere      oval

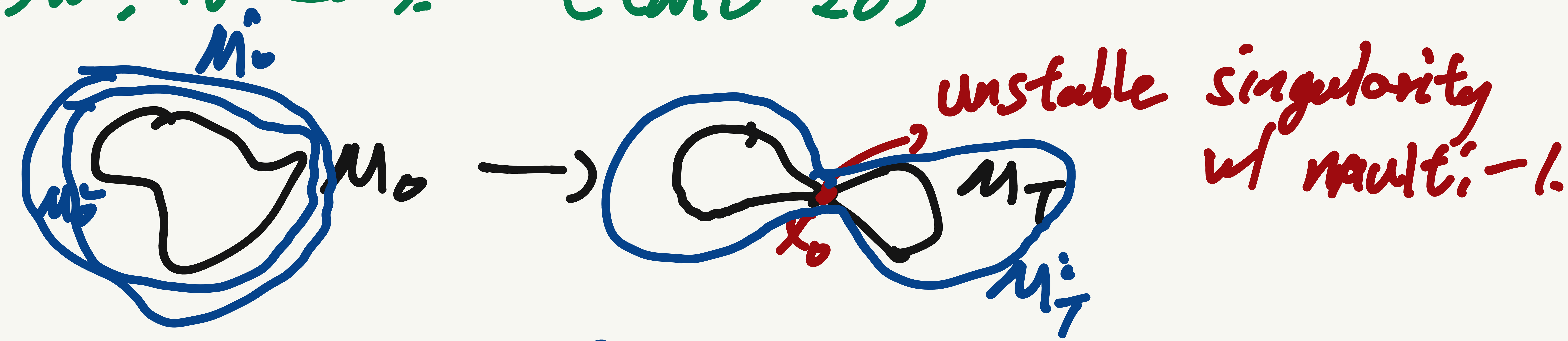
BC 18', 19') class. of conv. 2-convex. n-c. ancient flow

$\Rightarrow$  , 

Bowl.      cylinder

Avoidance principle at closed or asymptotically conical singularity w/ multi-one.

(BW, 16-20). (CCMS 20)




① Consider a seq $M_0^i \to M_0$ such that $M_0^i$ encloses $M_0^{i+1}$

$\Rightarrow M_t^i \cap M_t = \phi$ for $t < T$ by max. prin.

② Consider a subsequential limit $\lambda_i ( M^i_{\lambda_i^{-2}(t_0 - t)} - x_0 ) = \bar{M}_t$

Then, $\tilde{M}_t \cap \sqrt{-t}\,\Sigma = \phi$.

($\because$ $\sqrt{-t}\,\Sigma$ is the tangent flow of $M_t$)

Namely, $\tilde{M}_t$ is one-sided flow

$\Rightarrow$ RMCF $\hat{M}_\tau$ is the graph of $u : \Sigma \times (-\infty, T) \to \mathbb{R}$

such that $u > 0$

$\Rightarrow$ $\lim\limits_{\tau \to -\infty} e^{\lambda_1 \tau} u = a \varphi_1$ for $a > 0$

$\Rightarrow$ $u_\tau > 0 \Longleftrightarrow H - \frac{1}{2}\langle x, \nu \rangle > 0$

$\Longrightarrow$ By HW 20'. $\hat{M}_\tau$ has multi-1 round spherical

or cylindrical singularity.

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