Geometric inequalities and inverse mean curvature flow

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How much area a fixed lenghted curve could contain?

Theorem (Isoperimetric inequality)

Let $C = \partial\Omega$ be a closed and embedded curve in $\mathbb{R}^2$. Then

$$4\pi A \leq L^2$$

and the equality holds iff C is a round circle.

Long history with many different proofs.

Today we focus on a proof which uses the curvature flow and its implications on the concept of mass in general relativity.

Curve shortening flow

Each point on a curve moves with the velocity of its curvature vector.

More precisely, 1-parameter family of immersions

$$F : \mathbb{S}^1 \times [0, T] \to \mathbb{R}^2$$

is a solution to the CSF if

$$\frac{\partial}{\partial t} F(p, t) = \vec{k}(p, t).$$

Theorem (Gage-Hamilton '86)

Convex embedded curve shrinks to a point and becomes round.

Theorem (Grayson '87)

Embedded curve becomes convex in finite time.

Computer simulations

If $\vec{k} = k\vec{n}$ with the inner unit normal $\vec{n}$,

$$\partial_t A = -\int k ds = -2\pi$$

and by the first variation formula of area (length)

$$\partial_t L = -\int k^2 ds.$$

More generally, $M_t^n = \partial\Omega_t$ in $\mathbb{R}^{n+1}$ evolves by the velocity $S\vec{n}$,

$$\partial_t \mathrm{Vol}(\Omega_t) = -\int_{M_t} S dV_g$$

$$\partial_t |M_t^n| = -\int_{M_t} SH dV_g.$$

Proof of isoperimetric inequality

Let C_0 be given curve and C_t be the CSF which converges to a round point as $t \to t_{\max}$.

$$\partial_t(L^2 - 4\pi A) = -2L \int k^2 ds + 8\pi^2$$
$$\leq -2 \left(\int k ds \right)^2 + 8\pi = 0$$

Since $L^2 - 4\pi A \to 0$ as $t \to t_{\max}$ by Grayson,

$$L^2 \geq 4\pi A \quad \text{at} \quad t = 0$$

and the equality case easily follows.

Generalizations into two directions

- Non-Euclidean ambient space
- Higher dimensions

Theorem

For simply connected manifold (M^2, g) w/ Gaussian curvature $K \leq K_0$,

$$L^2 \geq 4\pi A - K_0 A^2$$

and the equality is attained for geodesic spheres which enclose constant curvature K_0 regions.

The more curved, the more area could be bounded. This suggests an idea that the isoperimetric constant or profile could be used to measure how much space is curved.

Higher dimensions

Theorem

Let M^n be the boundary of the region $\Omega \subset \mathbb{R}^{n+1}$. Then,

$$\mathrm{Vol}(\Omega) \le C_n |M^n|^{\frac{n+1}{n}}$$

and the equality holds iff Ω is a round ball.

Let $M_t^n = \partial\Omega_t \subset \mathbb{R}^{n+1}$ be a solution to the mean curvature flow. i.e. velocity $= H\vec{n}$, $H = \lambda_1 + \ldots + \lambda_n$.

$$\begin{aligned}\partial_t(C_n A^{\frac{n+1}{n}} - V) &= -\frac{n+1}{n} C_n A^{\frac{1}{n}} \int_{M_t} H^2 + \int_{M_t} H \\ &\le \left[1 - \frac{(n+1)C_n}{n} \frac{\int_{M_t} H}{A^{\frac{n-1}{n}}}\right] \int_{M_t} H\end{aligned}$$

Suppose we have the following:

1. $\int_M H \geq \frac{n}{(n+1)C_n}|M|^{\frac{n-1}{n}}$
2. $|M_t| \to 0$ and $\mathrm{Vol}(\Omega_t) \to 0$ as $t \to T_{\max}$.

Then we have the isoperimetric inequality

$$C_n|M_0|^{\frac{n+1}{n}} - \mathrm{Vol}(\Omega_0) \geq \lim_{t\to T_{\max}} C_n A^{\frac{n+1}{n}} - V = 0.$$

Note we have $\frac{n}{(n+1)C_n} = n|\partial B_1|^{\frac{1}{n}}$ and (1) is called the Minkowski inequality in $\mathbb{R}^{n+1}$.

Theorem (Minkowski 1903, Alexandrov '37 and others)

For convex hypersurfaces,

$$\int_M H \geq n|\partial B_1|^{\frac{1}{n}}|M|^{\frac{n-1}{n}}$$

with the equality iff round spheres.

- For mean convex star-shaped hypersurfaces, by Guan-Li '09
- For outward area minimizing hypersurfaces, by Huisken (unpublished)
- Open for general mean convex hypersurfaces

Theorem (Huisken '84)

Compact convex embedded hypersurface shrinks to a point and becomes round in finite time.

Another approach H^k-flow by F. Schulze ('08)

Consider the flow with the velocity $H^k\vec{n}$.

$$\partial_t(C_n A^{\frac{n+1}{n}} - V) = -\frac{n+1}{n} C_n A^{\frac{1}{n}} \int_{M_t} H^{k+1} + \int_{M_t} H^k$$

Meanwhile we play with Hölders

$$\int H^k \leq A^{\frac{1}{k+1}} \left(\int H^{k+1} \right)^{\frac{k}{k+1}}$$

and (if $k+1 \geq n$)

$$\left(\int H^n \right)^{\frac{1}{n}} \leq A^{\frac{1}{n} - \frac{1}{k+1}} \left(\int H^{k+1} \right)^{\frac{1}{k+1}}$$

imply

$$\int H^k \leq A^{\frac{1}{n}} \left(\int H^n \right)^{-\frac{1}{n}} \int H^{k+1}$$

$$\partial_t(C_n A^{\frac{n+1}{n}} - V) \le A^{\frac{1}{n}} \int H^{k+1} \left(\left(\int H^n \right)^{-\frac{1}{n}} - \frac{n+1}{n} C_n \right)$$

Theorem (Willmore inequality)

For immersed hypersurfaces

$$\int \left(\frac{H}{n} \right)^n \ge A(\partial B_1)$$

and the equality iff round spheres.

implies

$$\partial_t(C_n A^{\frac{n+1}{n}} - V) \le 0$$

Level set weak flow

Let $M_t = \partial\Omega_0 \subset \mathbb{R}^{n+1}$ be a solution to H^k-flow with $H > 0$.
If $u(x) :=$ the time t when M_t arrives at x, then

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = -|\nabla u|^{-\frac{1}{k}} & \text{on } \Omega_0 \\ u = 0 & \text{on } \partial\Omega_0 \end{cases}$$

- Weak formulation of flows defined through singularities
- Evans-Spruck and Chen-Giga-Goto at '91

Inverse mean curvature flow - definition

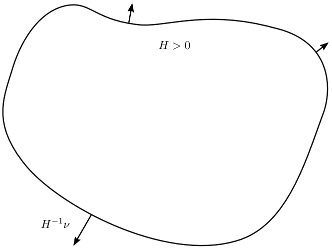
- A one-parameter family of mean convex embeddings $F(M^n,t) = M_t^n = \partial\Omega_t$

$$F : M^n \times [0,T] \to (\bar{M}^{n+1}, \bar{g})$$

is a solution to the inverse mean curvature flow if

$$\frac{\partial}{\partial t}F = \frac{\vec{\nu}}{H} \quad \text{on} \quad M^n \times [0,T].$$

Here $\vec{\nu}(p,t) :=$ outward unit normal.



Simple observations

- Expanding sphere solutions $\partial B_{R(t)} \subset \mathbb{R}^{n+1}$, $R(t) = e^{\frac{t}{n}} R(0)$

- $\partial_t d\text{vol} = d\text{vol} \to \partial_t |M_t| = |M_t|$, $|M_t| = e^t |M_0|$

- In $\mathbb{R}^{n+1}$, $M_t \to \lambda M_t$ produce another solution
  (unlike the mean curvature flow: $M_t \to \lambda M_{\frac{t}{\lambda^2}}$)

IMCF on compact hypersurfaces - background

- C. Gerhardt '90 and J. Urbas '90:
  If M_0 compact, smooth, star-shaped ($\langle F, \nu \rangle > 0$) and strictly mean convex, then smooth solution exists for $t \in (0, \infty)$ and converges to a spherical solution after a rescaling.

- For non star-shaped case, singularities may develop.
  e.g. mean convex torus in $\mathbb{R}^3$.

- G. Huisken, T. Ilmanen '97
  Defined a variational weak solution of the flow using level sets, which allows jumps of the surfaces. They use this to prove Riemannian Penrose inequality in General Relativity for a single black hole case.

The Geroch monotonicity formula

Geroch- the Hawking quasi-local mass of a connected 2-surface in a manifold of nonnegative scalar curvature is non-decreasing under the IMCF. For 2-surface M in $(\bar{M}^3, \bar{g})$,

$$m_H(M) := \frac{|M|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_M H^2\right).$$

Proof: Since $|M_t| = e^t |M_0|$, $\partial_t |M_t|^{1/2} = \frac{1}{2}|M_t|^{1/2}$

$$\partial_t m_H(M_t) = \frac{|M_t|^{1/2}}{(16\pi)^{3/2}} \left[\frac{1}{2}\left(16\pi - \int_M H^2\right) - \partial_t \int_{M_t} H^2\right]$$

Note $\partial_t H = -\Delta \frac{1}{H} - \frac{|A|^2}{H} - \frac{\overline{Ric}(\nu,\nu)}{H}$ and $\partial_t dvol = dvol$ imply

$$(*) \quad \partial_t \int_{M_t} H^2 = \int_{M_t} -2H\Delta \frac{1}{H} - 2|A|^2 - 2\overline{Ric}(\nu,\nu) + H^2.$$

Recall the Gauss equation

$$Rm_{ijkl} = \overline{Rm}_{ijkl} + (h_{ik}h_{jl} - h_{il}h_{jk})$$
$$\Rightarrow K = R_{1212} = \overline{Rm}_{1212} + \lambda_1\lambda_2 = \overline{Rm}_{1212} + \frac{1}{2}(H^2 - |A|^2).$$

In dim 3, $\frac{1}{2}\overline{R} = \overline{Ric}(\nu,\nu) + \overline{Rm}_{1212}$ and we get

$$-2\overline{Ric}(\nu,\nu) = -\overline{R} + 2K + |A|^2 - H^2.$$

$$
\begin{aligned}
\partial_t \int_{M_t} H^2 &= \int_{M_t} -2\frac{|\nabla H|^2}{H^2} - |A|^2 - \overline{R} + 2K \\
&= 4\pi\chi(M_t) + \int_{M_t} -2\frac{|\nabla H|^2}{H^2} - |A|^2 - \overline{R} \\
&\leq 8\pi - \int_{M_t} |A|^2 \quad (\chi(M_t) \leq 2 \text{ since } M_t \text{ is connected }) \\
&= 8\pi - \int_{M_t} \frac{1}{2}H^2 + \frac{1}{2}(\lambda_1 - \lambda_2)^2 \\
&\leq \frac{1}{2}\left(16\pi - \int_{M_t} H^2\right)
\end{aligned}
$$

$$
\Rightarrow \partial_t m_H(M_t) \geq 0.
$$

Isolated gravitating system in general relativity

Definition (Asymptotically flat manifold)

A complete manifold $(\bar{M}^3, \bar{g})$ is asymptotically flat (with order $\tau \in (1/2, 1])$ if there is a compact set $K \subset \bar{M}$ and a diffeomorphism

$$\phi : \mathbb{R}^3 - B_R(0) \to \bar{M}^3 - K$$

such that $\phi^* g = \delta_{ij} + \sigma_{ij}$ satisfies

$$|\sigma| + r|\partial\sigma| + r^2|\partial\partial\sigma| + r^3|\partial\partial\partial\sigma| \leq O(r^{-\tau})$$

Definition (ADM mass)

$$m_{ADM}(\bar{M}, \bar{g}) = \lim_{r\to\infty} \frac{1}{16\pi} \int_{\partial B_r(0)} (g_{ij,i} - g_{ii,j})\nu^j dV_{g_{euc}}$$

Schwarzschild metric

Schwarzschild metric of mass $m > 0$ is

$$(\mathbb{R}^3 - \{0\}, \big(1 + \frac{m}{2|x|}\big)^4 \delta_{ij})$$

and note that

- $r = \frac{m}{2}$ is a totally geodesic sphere and is called (apparent) horizon
- $A(\{r = \frac{m}{2}\}) = 16\pi m^2$
- no compact minimal surface in $\{r > \frac{m}{2}\}$
- $m_{ADM} = m$.

Theorem (Riemannian Penrose inequality)

Let $(\bar{M}^3, \bar{g})$ be an asymptotically flat mfd with nonnegative scalar curvature with minimal boundary $M_0 = \partial \bar{M}$ and no other compact minimal surface in $\bar{M}$. Then

$$m_{ADM} \geq \sqrt{\frac{|M_0|}{16\pi}}.$$

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Heuristic Proof: Let M_t IMCF from M_0. By the monotonicity

$$m_H(M_t) \geq m_H(M_0) = \frac{|M_0|^{1/2}}{(16\pi)^{3/2}} \left(16\pi - \int_{M_0} H^2\right) = \sqrt{\frac{|M_0|}{16\pi}}.$$

It is known that the Hawking mass of sufficiently round (coordinate) spheres at infinity approaches to the ADM mass. If we show M_t becomes sufficiently round spheres at infinity,

$$m_{ADM} = \lim_{t\to\infty} m_H(M_t) \geq \sqrt{\frac{|M_0|}{16\pi}}.$$

Properties of weak solution

For $M_0 = \partial\Omega_0$,

$$\begin{cases} \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = |\nabla u| & \text{on } \Omega_0^c \\ u = 0 & \text{on } \partial\Omega_0 \end{cases}$$

and $M_t = \partial\{u < t\}$.

- Ω_t is outward area minimizing in the sense that

$$\Omega_t \subset \Omega' \Longrightarrow |\partial\Omega_t| \leq |\partial\Omega|$$

- $|M_t| = e^t |M_0|$

Isoperimetric profile and m_{ADM}.

For Schwarzschild of mass m, $(\mathbb{R}^3 - \{0\}, \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij})$, the isoperimetric profile is the function

$$\begin{aligned} \phi_m : [16\pi m^2, \infty) &\longrightarrow \mathbb{R} \\ |\partial B_r(0)| &\longmapsto \mathrm{Vol}(B_r(0) - B_{\frac{m}{2}}(0)) \end{aligned}$$

Key observation

$$\phi_m(A) = \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}} + \frac{1}{2} m A + O(A^{\frac{1}{2}}) \text{ as } A \to \infty.$$

This motivates the following

Definition

$$m_{iso}(\bar{M}^3, \bar{g}) = \limsup_{|\partial\Omega|\to\infty} \frac{2}{|\partial\Omega|}(\mathrm{Vol}(\Omega) - \frac{1}{6\sqrt{\pi}}|\partial\Omega|^{\frac{3}{2}})$$

Theorem (Huisken, unpublished)

$$m_{iso}(\bar{M}, \bar{g}) = m_{ADM}(\bar{M}, \bar{g})$$

The proof uses both MCF and IMCF.

Key tool

Theorem (J.L.Jauregui-D.A.Lee, Huisken)

Under the MCF M_t, if $m_H(M_t) \leq m'$, then

$$\phi_{m'}(A(t)) - V(t) \text{ is decreasing in } t.$$

Recall in isoperimetric inequality,

$$\left(\int_M H^2\right)^{1/2} \geq \Lambda \text{ for all } M \quad \Longrightarrow \quad \partial_t \left(\frac{2}{3}\frac{A^{\frac{3}{2}}}{\Lambda} - V\right) \leq 0.$$

$$\begin{aligned} V(0) - \frac{A(0)}{6\sqrt{\pi}} &\leq \phi_{m'}(A(0)) - \frac{A(0)}{6\sqrt{\pi}} - [\phi_{m'}(A(t)) - V(t)] \\ &\leq \phi_{m'}(A(0)) - \frac{A(0)}{6\sqrt{\pi}} + v \end{aligned}$$

implies

$$m_{iso} \leq m'$$

Other applications

By similar arguments, the flow (classical or weak) has been used in the proofs of other geometric inequalities as well.

- Guan-Li Minkowski ineq for non-convex hypersurface in $\mathbb{R}^{n+1}$
- Brendle-Hung-Wang Minkowski ineq for Anti-deSitter Schwarzschild manifold
- Lee-Neves Penrose type ineq for asymptotically locally hyperbolic space
- De Lima-Girão, Ge-Wang-Wu Alexandrov-Fenchel ineq for different ambient spaces
- Bray-Neves another important application to the classification of prime 3-manifolds with Yamabe invariant greater than $\mathbb{RP}^3$.

Regularity of compact star-shaped solutions

Theorem (Huisken-Ilmanen '08, H^{-1} estimate)

Suppose $F : M^n \times [0,T] \to \mathbb{R}^{n+1}$ *is a smooth star-shaped IMCF such that on* M_0^n

$$0 < R_1 \le \langle F, \nu \rangle \le R_2.$$

Then there is $c_n > 0$ *such that for* $t > 0$

$$\frac{1}{H} \le c_n \left(\frac{R_2}{R_1} \right) \max\Big(1, \frac{1}{t^{1/2}}\Big) \, |M_0^n|^{\frac{1}{n}} \, e^{\frac{t}{n}}.$$

- Estimate does not depend on initial bound on H^{-1} or $|A|^2$
- Uses only initial bounds of the support function $\langle F, \nu \rangle$

Theorem (Smoczyk ($n = 2$) '00, Huisken-Ilmanen '08)

Let $F : M^n \times [0, T] \to \mathbb{R}^{n+1}$ be a smooth compact sol of IMCF.

$$\text{If } 0 < H_0 \le H \le H_1, \text{ then } |A|^2 \le c_n \frac{H_1^2}{H_0} \frac{1}{t}.$$

- $\frac{\partial H}{\partial t} = \frac{\Delta H}{H^2} - \frac{2|\nabla H|^2}{H^3} - \frac{|A|^2}{H} = \nabla \cdot (H^{-2} \nabla H) - \frac{|A|^2}{H}$
- Huisken-Ilmanen showed the existence of smooth sol when M_0 is C^1, compact, star-shaped and has bounded non-negative (weak) mean curvature.

Non-compact solutions in $\mathbb{R}^{n+1}$

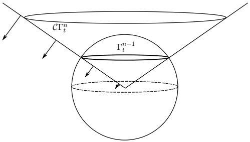
An important observation

Example

If $\Gamma_t^{n-1} \subset \mathbb{S}^n$ is a IMCF, then the cones generated by Γ_t

$$\mathcal{C}\Gamma_t = \{rx \in \mathbb{R}^{n+1} \,:\, x \in \Gamma_t,\, r \geq 0\} \subset \mathbb{R}^{n+1}$$

is a IMCF which is smooth except the origin.



In simplest case, if $\mathcal{C}\Gamma_0 := \{(x, \tan\theta_0\,|x|)\}$ is a round cone, then

$$\mathcal{C}\Gamma_t := \{(x, \tan\theta(t)\,|x|)\} \text{ for } t \in [0, T^*) \text{ with } T^* = \ln\frac{\cos^{n-1} 0}{\cos^{n-1}\theta_0}.$$

Convex solutions in $\mathbb{S}^n$ and dualitiy

Theorem (Gerhardt '15, Makowski-Scheuer '16)

Let $\Gamma_0 \subset \mathbb{S}^n$ be smooth, strictly convex ($h_{ij} > 0$). There exist a unique smooth IMCF, Γ_t, for $t \in [0, T)$ and the Γ_t converges to an equator as $t \to T$. (Note $T = \ln |\mathbb{S}^{n-1}| - \ln |\Gamma_0^{n-1}|$)

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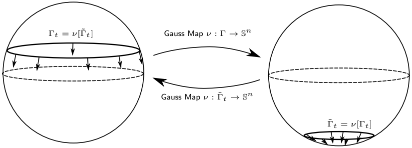
Gerhardt - for strictly convex solutions of $F(\lambda_i)$ flow in $\mathbb{S}^n$

$$\begin{aligned} &\Gamma_t \subset \mathbb{S}^n \\ &x_t = F(\lambda_i)\nu \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} &\tilde{\Gamma}_t = \nu[\Gamma_t] \subset \mathbb{S}^n \\ &\tilde{x}_t = -\tilde{F}(\tilde{\lambda}_i)\tilde{\nu} \end{aligned}$$

with $\tilde{F}(\tilde{\lambda}_i) = F(\tilde{\lambda}_i^{-1})$.



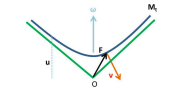
Non-compact solutions in $\mathbb{R}^{n+1}$ - first result

Theorem (Daskalopoulos-Huisken '17)

For $n \geq 2$, let $M_0 \subset \mathbb{R}^{n+1}$ be C^2 convex entire graph $x_{n+1} = u_0(x)$, $x \in \mathbb{R}^n$ with:

1. $\alpha_0 |x| < u_0(x) < \alpha_0 |x| + \kappa$ *for some* $\alpha_0 > 0$ *and* $\kappa > 0$
2. $0 < c_0 < H(\langle F, e_{n+1}\rangle + 1) < C_0$.

Then, there exists unique smooth IMCF M_t for $0 < t < T$ with $T = (n-1)\ln\sqrt{1+\alpha_0^2}$ and M_t becomes flat as $t \to T$.



Solution no longer has a uniform lower bound of H, and hence local estimate of H^{-1} was essential. Long time existence was a hard part as $H|F|$ (or $H\langle F, e_{n+1}\rangle$) vanishes as $t \to T$ and they had to capture this behavior.

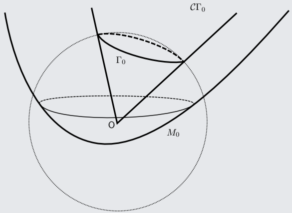
Non-compact solutions in $\mathbb{R}^{n+1}$ - main existence result

Definition (Tangent cone at infinity)

Let $M_0 = \partial \hat{M}_0$, $\hat{M}_0 \subset \mathbb{R}^{n+1}$ non-compact convex set (containing the origin). Then

$$\cap_{\epsilon>0} \epsilon \hat{M}_0 = \mathcal{C}\hat{\Gamma}_0 \text{ for some convex set } \hat{\Gamma}_0 \subset \mathbb{S}^n.$$

We call it the tangent cone of $\hat{M}_0$ at infinity. $\Gamma_0 := \partial \hat{\Gamma}_0$ in $\mathbb{S}^n$ and we call $\mathcal{C}\Gamma_0$ the tangent cone of $M_0 := \partial \hat{M}_0$ at infinity.



Motivated from the previous observation and the scaling property of the flow, we prove

Theorem (C.-Daskalopoulos '18)

For convex $M_0 = \partial \hat{M}_0 \subset \mathbb{R}^{n+1}$, IMCF M_t exists for $t \in (0, T)$.

- *Using $P(\hat{\Gamma}_0) :=$ the perimeter of $\hat{\Gamma}_0$ in $\mathbb{S}^n$,*

  $$T = \ln |\mathbb{S}^{n-1}| - \ln P(\hat{\Gamma}_0) \in [0, \infty]$$

  and it is maximal. i.e. no smooth solution exists for $t > T$.
- *Let $\mathcal{C}\Gamma_t :=$ the tangent cone of M_t at infinity. Then $\Gamma_t \subset \mathbb{S}^n$ is IMCF in $\mathbb{S}^n$ becoming flat as $t \to T$.*

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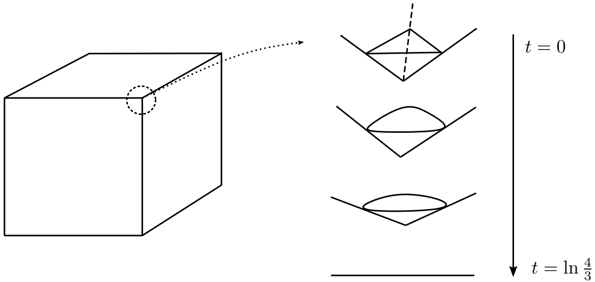
Note $\hat{\Gamma}_0$ can be any convex set in $\mathbb{S}^n$ possibly be degenerate (e.g. line, point, etc) and

$$P(\hat{\Gamma}_0) = \begin{cases} |\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has non-empty interior in } \mathbb{S}^n \\ 2|\Gamma_0| & \text{if } \hat{\Gamma}_0 \text{ has empty interior in } \mathbb{S}^n. \end{cases}$$

Evolution of Singularity

Theorem (C.- Hung '18)

If $0 \in M_0$ is singular in the sense that the (blow-up) tangent cone $T_0 M_0 \neq \mathbb{R}^n$, then $0 \in M_t$ until the tangent cone becomes flat and $T_0 M_t$ evolves by IMCF.



Theorem (C.-Daskalopoulos '18 + C.-Hung '18)

For an arbitrary convex $M_0 = \partial \hat{M}_0 \subset \mathbb{R}^{n+1}$, a convex smooth solution exists if and only if the density of $M_0 \equiv 1$.

$$\textit{i.e. } \Theta_0(p) = \lim_{r \to 0} \frac{|B_r(p) \cap M_0|_n}{\omega_n r^n} = 1 \textit{ for all } p \in M_0.$$

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When $\Theta_0(p) = 1$? $\Theta_0(p) = \frac{|T_pM_0 \cap \mathbb{S}^n|_{n-1}}{|\mathbb{S}^{n-1}|_{n-1}}$ with $T_pM_0 :=$ the tangent cone at $p \in M_0$.

If we denote $\Gamma_0(p) = T_pM_0 \cap \mathbb{S}^n$ and $\Gamma_0(p) = \partial \hat{\Gamma}_0(p)$,

$$\theta_0(p) = 1 \text{ iff } P(\hat{\Gamma}_0(p)) = |\mathbb{S}^{n-1}|.$$

This also relates the case $T = 0$ (non-existence of solution).

Lemma (C.-Hung)

If $\hat{\Gamma}_0 \subset \mathbb{S}^n$ convex and $P(\hat{\Gamma}_0) = |\mathbb{S}^{n-1}|$, then $\hat{\Gamma}_0$ is either a hemisphere or a wedge [we call it a (hard shell) taco].

i.e. $\hat{W}_{\theta_0} = \mathbb{S}^n \cap \big(\{(r\sin\theta, r\cos\theta) \,:\, \theta \in [0,\theta_0],\ \text{and } r > 0\} \times \mathbb{R}^{n-1}\big)$

for some $\theta_0 \in [0,\pi]$ up to an isometry of $\mathbb{S}^n$.

- Smooth flow exists iff all initial singularities are tacos.
- No solution exists iff asymptotic cone (cone at infinity) is a taco. ($\Rightarrow$ no 1-dimensional convex non-compact IMCF exists)

Theorem (C.-Daskalopoulos, main apriori estimate)

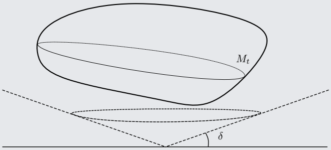
Let $F:\Sigma\times[0,T]\to\mathbb{R}^{n+1}$, $n\geq 2$, *be a convex compact IMCF and suppose there exist* $\delta\in(0,\pi/2)$ *and a fixed unit vector* $\omega\in\mathbb{R}^{n+1}$ *for which*

$$\langle F,\omega\rangle \geq |F|\sin\delta \quad \text{on } \Sigma\times[0,T].$$

Then

$$\frac{1}{H|F|}\leq C\left(1+\frac{1}{t^{1/2}}\right) \quad \text{on} \quad \Sigma\times[0,T]$$

for a constant $C=C(\delta)>0$.



Theorem (C.-Daskalopoulos (c.f. Huisken-Ilmanen '08))

For M_t compact star-shaped IMCF with

$$0 < R_1 \leq \langle F, \nu \rangle \leq R_2 \quad \text{on } M_0$$

(the same condition of Huisken-Ilmanen), we have

$$\frac{1}{H} \leq c_n \left(\frac{R_2}{R_1} \right) \left(1 + \frac{1}{t^{1/2}} \right) R_2 \, e^{\frac{t}{n}},$$

(only $|M_0|^{1/n}$ has been replaced by R_2.)

Thank you for your attention

Useful reference (for m_{iso} and Riemannian Penrose inequality)

- Huisken's talk at IAS (available at youtube)
- J.L.Jauregui-D.A.Lee '16
- Bray's AMS notice