참고자료 (모두 온라인에서 찾을 수 있음)

(1) B. Eynard. Lectures on compact Riemann surfaces. Doctoral. Paris-Saclay, France. 2018. cel-02013559

이 자료의 1장과 2장에서 강의록의 틀을 가져왔습니다. 강의 내용을 좀 더 자세히 살펴보고자 한다면 이 자료를 찾아보시면 됩니다.

(2) Riemann surfaces, dr C. Teleman (Lent Term 2003)

대학에서 강의한 자료인 것 같은데, 일독을 권합니다. 자세한 내용은 종종 빠져있지만 직관적인 설명이 손그림과 함께 흥미롭게 제시되어 있습니다.

(3) A minicourse on moduli of curves, Eduard Looijenga (Lectures given at the School on algberaic Geometry, Trieste, 1999)

Riemann surface의 moduli count $3g - 3$를 다양한 관점에서 설명합니다.

(4) David D. Ben-Zvi, Moduli spaces

거의 수식이 없는 글로, moduli의 개념에 대해 소개합니다. moduli space의 대표적인 예로 moduli of elliptic curves와 Teichmüller space를 다루고 있기 때문에 본 강의와 김현규 교수님 강의에 좋은 참고자료가 될 것 같습니다.

(5) Cohomological obstructions for Mittag–Leffler Problems, Mateus Schmidt, arXiv:2010.11812v1 [math.CV] 22 Oct 2020

리만면 연구를 Mittag–Leffler problem을 일반화한다는 것과 같은 것으로 이해할 수 있습니다. 이런 관점을 정확히 서술하기 위한 여러 종류의 cohomology 이론과 특히 sheaf cohomology가 상세히 설명되어 있습니다.

There are three doors to access to the object called a Riemann surface: differential geometry, algebraic geometry, and complex analysis. Historically the concept of Riemann surface emerged as a geometric object to understand and solve the problems in these areas. An access from algebraic geometry has computational advantages, but it may hide the geometric intuitions behind the computations. An access from differential geometry is quite a natural approach, and I hope we can have a lecture series in this prospect in the next years. In this lecture, I'll adopt complex analysis as the main tool for explanation, because it captures the point with a small amount of machinery.

A *Riemann surface* is a smooth surface whose transition maps $\{\psi_{U\to V}\}$ are biholomorphic. In other words, it is a surface equipped with a complex structure. In this talk, we will consider the *compact* Riemann surfaces only.

Every Riemann surface is orientable, because the complex structure provides the orientation in each chart: $v \mapsto \sqrt{-1}v$. Hence the only topological invariant is the genus g (or the Euler characteristic $\chi = 2 - 2g$).

As I understand, the original definition of Riemann surfaces was given in terms of conformality. A $\mathcal{C}^1$-map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is conformal (angle-preserving) and preserves the orientation if and only if it satisfies the Cauchy-Riemann equations. In this sense, a Riemann surface is an orientable surface equipped with a notion of angles. (The metric $(v, v) = \sqrt{||v||}$ is not preserved, but we can measure the angle $\dfrac{(u, v)}{||u||\,||v||}$ on a Riemann surface). This gives hint to the fact that complex tori $\mathbb{C}/\Lambda$ may have different structures of Riemann surfaces as the lattices are deformed.

Find pictures from David Xianfeng Gu's gallery:
https://www3.cs.stonybrook.edu/ gu/gallery/RiemannSurface/index.html

The Riemann sphere S^2 is the Riemann surface of genus 0. It is the sphere $X^2+Y^2+Z^2 = 1$ equipped with two charts given by the stereographic projections ϕ_1 (from the north pole) and ϕ_2 (from the south pole):

$$\phi_1(X, Y, Z) = \frac{X + iY}{1 + Z} \quad \text{and} \quad \phi_2(X, Y, Z) = \frac{X - iY}{1 - Z}.$$

Note that the transition map $\psi_{1\to 2} = \phi_2 \circ \phi_1^{-1} = 1/z$ is biholomorphic:

$$\left(\frac{X + iY}{1 + Z}\right)^{-1} = \frac{1 + Z}{X + iY} = \frac{(1 + Z)(X - iY)}{X^2 + Y^2} = \frac{X - iY}{1 - Z}$$

There is no (nonconstant) holomorphic function on the whole Riemann sphere, due to the Liouville's theorem. Hence we should consider the meromorphic functions instead. Riemann sphere can be regarded as a mathematical object/construct on which we can run the complex analysis (on $\mathbb{C}$), observing the behaviour near ∞.

Every meromorphic function f on the Riemann sphere is a rational function $\dfrac{P(z)}{Q(z)}$: After killing the poles of f inside $\mathbb{C}$ by multiplying a suitable polynomial, the function $f(z)\cdot\prod_{i=1}^{k}(z-z_i)^{m_i}$ is an entire holomorphic function which may have a zero/pole at ∞. It must be a rational function.

A torus $T_\tau = \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}),\ \tau \in \mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, has a structure of Riemann surface of genus 1. The transition maps between two charts are given by the translations, which are obviously biholomorphic. Again, there is no (nonconstant) holomorphic function on the Riemann surface T_τ, because every doubly periodic holomorphic function is constant. Hence we should consider the doubly periodic meromorphic functions on $\mathbb{C}$. They are the meromorphic functions ϕ on $\mathbb{C}^*$ such that $\phi(qw) = \phi(w)$, where $q = \exp(2\pi i\tau)$. Indeed, $\mathbb{C}/\mathbb{Z} \cong \mathbb{C}^*$ and a doubly periodic function on $\mathbb{C}$ can be regarded as a function $\phi : \mathbb{C}^*/\langle q\rangle \to \mathbb{C}$. For example, the function

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{w\,\in\,\mathbb{Z}+\tau\mathbb{Z}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2}\right)$$

is a meromorphic function on T_τ (called the *Weierstrass $\mathcal{P}$-function*), which has a double pole at $z = 0$.

Notably, there is a *moduli* of Riemann surfaces of genus 1, produced by the lattice point τ. For example, the Riemann surfaces $\{T_{ib} = \mathbb{C}/(\mathbb{Z} + ib\mathbb{Z}) : 1 < b\}$ are not biholomorphic to each other. There is a 1-(complex)-parameter family of complex analyses over a torus. (In contrast, there is a unique complex analysis on S^2.)

Right after Riemann defined the objects corresponding to "Riemann surfaces", he computed the *moduli*(= number of parameters) of Riemann surfaces. For higher genus $g > 1$, "Riemann's count" says there are $3g-3$ moduli of Riemann surfaces of genus g. We will see this number later in the lecture on Teichmüller space. This numbers are given as the first Čech cohomology group of the holomorphic tangent bundle:

$$H^1(S, \Theta_S) \; = \; \begin{cases} 0, & g = 0 \\ 1, & g = 1 \\ 3g-3, & g > 1 \end{cases}$$

A concrete way to get Riemann surfaces is provided by the algebraic equations. For a polynomial function $P(x, y) \in \mathbb{C}[x, y]$, put

$$\Sigma = \{(x, y) \; : \; P(x, y) = 0\} \; \subset \; \mathbb{C}^2.$$

This is a "branched covering" of $\mathbb{C}$ given by the projection $(x, y) \mapsto x$. On each neighborhood of $(x_0, y_0) \in \Sigma$ such that $\dfrac{\partial P}{\partial y}(x_0, y_0) \neq 0$, we can use x as the coordinate of the "complex algebraic curve". Let

$$\Sigma_{sing} = \{(x, y) \; : \; P(x, y) = 0 \text{ and } \frac{\partial P}{\partial y}(x, y) = 0\}.$$

The x-coordinates of Σ_{sing} are precisely the solutions to the equation

$$0 = \text{Discriminant}(P(x, \cdot)) \; = \; \text{Resultant}\left(P(x, \cdot), \; \frac{\partial P}{\partial y}(x, \cdot)\right),$$

which is a finite set of isolated points. We can compactify Σ by adding the points at infinity: Embed Σ into $\mathbb{CP}^2$ and take the closure("projectivization"). The resulting object $\overline{\Sigma}$ is a complex manifold outside a finite number of singularities at the points Σ_{sing} and possibly at the points of infinity($\overline{\Sigma} \setminus \Sigma$). We need to "resolve" the singularities of $\overline{\Sigma}$, which can be cooked according to an algebraic recipe. Here are samples:

Sample 1 $\Sigma : \; x^2 + y^2 = 1 \; \subset \mathbb{C}^2$.

The projectivization $\overline{\Sigma}$ is given by $X^2 + Y^2 = Z^2$ in $\mathbb{CP}^2$ and the points at infinity are:

$$\overline{\Sigma} \setminus \Sigma \; = \; \{(i : 1 : 0), \; (-i : 1 : 0)\}.$$

It can be checked that these two points are smooth points of $\overline{\Sigma}$ and the surface $\overline{\Sigma}$ is biholomorphic to the Riemann sphere by the Pythagorean triple formula $\Phi : \mathbb{C} \to \overline{\Sigma}$:

$$\Phi(z) = \left(\frac{2z}{1+z^2}, \; \frac{1-z^2}{1+z^2}\right).$$

This should be understood as the map $\Phi : \mathbb{C} \to \Sigma$ for $z \neq \pm i$ and

$$\Phi(\pm i) = (2z : 1 - z^2 : 1 + z^2)\Big|_{z=\pm i} = (\pm i : 1 : 0).$$

(The inverse map is given by $\Phi^{-1}(X : Y : Z) = \dfrac{X}{Y + Z}$.)

Sample 2 $\Sigma :\ y = x^3\ \subset \mathbb{C}^2$.

The projectivization $\overline{\Sigma}$ is given by $YZ^2 = X^3$ in $\mathbb{CP}^2$ with one point at infinity:

$$\overline{\Sigma} \setminus \Sigma\ =\ \{\infty := (0 : 1 : 0)\}.$$

Near ∞, the surface looks like a cusp $C : z^2 = x^3$. To resolve this singularity, embed this piece C into $\mathbb{C}^3$ and "blow up" by adding the "slope" variable:

$$Bl_\infty C\ =\ \{(x, z, w) \in \mathbb{C}^3\ :\ z^2 = x^3,\ xw = z\}.$$

This is union of two (complex) curves: $Bl_\infty C = C_1 \cup C_2$, where C_1 is projected down to C and C_2 is the w-axis. Since C_1 is parameterized by $(x = t^2,\ z = t^3,\ w = t)$, Since the smooth curve C_1 is isomorphic to C outside $(x, z) = (0, 0, 0)$, we say that C_1 is a desingularization of C. (The projection map $C_1 \to C$ is bijective, but it is not "biholomorphic" because C is not smooth at the origin.) The point $(0, 0) \in C$ is called a cusp.

Now the resolution of singularity $\widetilde{\Sigma}$ of $\overline{\Sigma}$ is obtained by replacing the piece $C \subset \overline{\Sigma}$ by C_1. It is again biholomorphic to the Riemann sphere via $z \mapsto (z, z^3)$.

More generally, starting from any graph curve $\Sigma :\ \ y = Q(x)$, we can resolve $\overline{\Sigma} \subset \mathbb{CP}^2$ to get a Riemann surface $\widetilde{\Sigma}$ which is biholomorphic to the Riemann sphere.

Sample 3 To get a Riemann surface of genus $g \geq 1$ from algebraic equation, we need to start from a polynomial $P(x, y)$ of degree ≥ 3. Let Σ be the plane curve associated to $P(x, y) = y^2 - x^3 + x$.

Note that $\Sigma_{sing} = \{(0, 0)\}$, but at this point $\frac{\partial P}{\partial x}(0, 0) = (1, 0)$. So we can use y as the local coordinate near $(0, 0)$. The compactified curve is given by $Y^2 Z - X^3 + XZ^3 = 0$ and this has one point at infinity: $\infty = (0 : 1 : 0)$.

Near ∞, this curve looks like $z - x^3 + xz^3 = 0$, which is smooth at $(0,0)$. Therefore, $\widetilde{\Sigma} = \overline{\Sigma}$ in this case.

What is the genus of $\widetilde{\Sigma}$? Let us briefly introduce the classical construction of $\widetilde{\Sigma}$ from the multi-valued function $f(x) = \sqrt{x^3 - x}$. First, the projection map $(x, y) \mapsto y$ is well-defined on Σ. In other words, the function $f(x) = \sqrt{x^3 - x}$ is well-defined and it extends to a meromorphic function on $\overline{\Sigma}$, or a holomorphic map $\overline{\Sigma} \to \mathbb{CP}^1$. Recall that this is not single-valued over $\mathbb{C}$ (or $\mathbb{CP}^1$). To explain more details of this phenomenon, observe that the projection map $\pi(x, y) = x$ makes $\overline{\Sigma}$ a double (branched) covering of $\mathbb{CP}^1$, with four branch points $\pm 1, 0$, and ∞. $\cdots\cdots$

Sample 4 Now we start from the polynomial: $P(x, y) = y + x^3 + x^2 y$. Note that the curve $\Sigma = \{P(x, y) = 0\}$ is smooth. But its projectivization is given by $YZ^2 + X^3 + X^2 Y = 0$, which has a point at infinity $\infty = (0 : 1 : 0)$. Near the point $(x, z) = (0, 0)$, this looks like $C : z^2 + x^3 + x^2 = 0$. It is singular at $(0, 0)$, so we blow it up to get:

$$Bl_\infty C \;=\; \{(x, z, w) \in \mathbb{C}^3 \;:\; z^2 + x^3 + x^2,\; xw = z\}.$$

Let $C_1 = Bl_\infty C \setminus w$-axis. Then along the curve C_1, we have

$$\lim_{(x,z)\to(0,0)} w^2 \;=\; \lim_{(x,z)\to(0,0)} \frac{-x^3 - x^2}{x^2} = -1.$$

Hence in the limit $(x, z) \to (0, 0)$, we have $w \to \pm i$. We see that in this case the projection $\pi : C_1 \to C$ is bijective outside w-axis, and $\pi^{-1}(0, 0) = (0, 0, \pm i)$. This is a resolution of a nodal singularity.

A polynomial $P(x, y) = y^2 - \prod_{i=1}^{d}(x - x_i)$ for distinct x_i's yields a Riemann surface of genus $g = \left\lfloor \dfrac{d-1}{2} \right\rfloor$. These Riemann surfaces are called the *hyperelliptic curves*, named after the elliptic curves which correspond to the $g = 1$ case ($d = 3, 4$). Similarly, we can associate a Riemann surface to arbitrary polynomial $P(x, y)$. In fact, the converse is also true:

Every compact Riemann surface can be algebraically immersed into $\mathbb{CP}^2$, *with at most simple nodal points.*

In this sense, the compact Riemann surface (in differential geometry or complex analysis) is the same as the algebraic curve (in algebraic geometry).

As we have seen, the function theory on a Riemann surface S and the geometry of S are closely related. First let's establish a general fact which we already observed for $g = 0$ and 1.

Theorem 1 *There is no nonconstant (global) holomorphic function on S.*

Proof. Let f be a holomorphic function. Since S is compact, $|f(z)|$ has maximum at some point, say $p \in S$. If we choose a chart $U \ni p$, the holomorphic function $f_U : U \to \mathbb{C}$ cannot attain the maximum absolute value inside U, unless it is constant (maximum principle). Therefore, f_U is constant and eventually f is constant by the identity theorem. □

Hence the space $\mathcal{O}(S)$ of holomorphic functions on S is nothing but $\mathbb{C}$. (There is no function theory on $\mathcal{O}(S)$.) And we should consider the following objects:

- $\mathcal{M}(S)$: meromorphic functions on S
- $\mathcal{O}^1(S)$: holomorphic forms on S
- $\mathcal{M}^1(S)$: meromorphic forms on S

A meromorphic function $f \in \mathcal{M}(S)$ is nothing but a holomorphic map $S \to \mathbb{CP}^1$. It is not at all obvious if there is any nonconstant meromorphic function on every Riemann surface.[1] In fact, there are plenty of them, but they are subject to a strong constraints as we will see.

A holomorphic(meromorphic) form on S is a collection of 1-forms $\{f_U(z)dz\}$ for some holomorphic(meromorphic) functions f_U on each chart $U \subset S$ such that for each intersection $U \cap V$:

$$f_U(z)dz = f_V(w)dw \quad (w = \phi_{UV}(z)).$$

Equivalently, a form on S is the collection of local functions $\{f_U\}$ satisfying

$$f_U(z) = (f_V \circ \phi_{UV}(z)) \cdot \phi'_{UV}(z).$$

[1] This is okay if we admit the fact that every Riemnan surface can be algebraically defined, because the curve $P(x, y) = 0$ has a projection map $(x, y) \mapsto x$.

More generally, we consider the kth order holomorphic(meromorphic) forms in $\mathcal{O}^k(S)$ $(or in \mathcal{M}^k(S))$ given by a collection of $\{f_U(z)dz^k\}$ satisfying

$$f_U = (f_V \circ \phi_{UV}) \cdot \phi'_{UV}(z)^k.$$

For example, the function $f(z) = z$ on the Riemann sphere is a meromorphic function with a simple pole at ∞, and the form $\omega(z) = dz$ is a meromorphic form which has a double pole at ∞: for $z' = 1/z$,

$$\omega(z) = dz = d(1/z') = \frac{-1}{(z')^2}dz'.$$

This shows that there is no holomorphic form on $\mathbb{CP}^1$: a form $f(z)dz$ in one chart is transformed to $-\frac{1}{(z')^2}f(\frac{1}{z'})dz'$ in another chart.

The *order* of a function $f = \{f_U\}$ (or a form $\omega = \{f_U dz\}$) at $p \in S$ is the order of f_U for a chart $U \ni p$. It is written as $\text{ord}_p f$ (or $\text{ord}_p \omega$). the order at p is zero if and only if f_U has neither a zero or a pole at p. Given a form $\omega \in \mathcal{M}(S)$, the *residue* of ω at p is $\text{Res}_p \omega = c_{-1}$, where

$$\omega = f_U(z)dz, \quad f_U(z) = \sum_{j=-1}^{-d} c_j z^j + \text{analytic}.$$

In fact, $\text{Res}_p f$ of a meromorphic function $f(z)$ on $\mathbb{C}$ is equal to $\text{Res}_p \omega$ of the form $\omega = f(z)dz$ on the Riemann sphere. Cauchy's theorem says that

$$\text{Res}_p \omega = \frac{1}{2\pi i} \int_{C_p} \omega,$$

where C_p is a small anti-clockwise contour around p.

Exercise The order $\text{ord}_p f$ and the residue $\text{Res}_p \omega$ are independent of the choice of the chart and coordinates. On the other hand, the coefficients c_j for $j \leq -2$ do depend on the choice.

Theorem 2 *For $\omega \in \mathcal{M}^1(S)$, the total residue is zero:*

$$\sum_p \text{Res}_p(\omega) = 0.$$

Proof. The surface S can be polygonalized: there is a graph on S whose faces are polygons entirely contained in a chart such that each pole of ω lie inside some polygon. The sum of residues is the sum of the integral along each polygons, which cancels out each other. $\square$

Corollary 3 *There is no $\omega \in \mathcal{M}^1(S)$ with only one simple pole.*

Given a nonzero $f \in \mathcal{M}(S)$, the *divisor of* f is defined by

$$(f) = \sum_{p \in S} (\mathrm{ord}_p f) p.$$

This is a formal $\mathbb{Z}$-linear combination of points of S. Similarly, given a nonzero $\omega \in \mathcal{M}^1(S)$, the *divisor of* ω is

$$(\omega) = \sum_{p \in S} (\mathrm{ord}_p \omega) p.$$

Each nonzero meromorphic function/form has only finitely many poles and zeros. (Why?)

Corollary 4 *For $g \geq 1$, there is no $f \in \mathcal{M}(S)$ with only one simple pole.*

Proof. We use the fact that there is a nonzero holomorphic form $\omega \in \mathcal{O}^1(S)$ for $g \geq 1$.[2] If $\mathrm{ord}_p(\omega) = k$, then $f^{k+1}\omega$ is a meromorphic form which has only one simple pole (at p), which is a contradiction. $\square$

The Riemann–Roch formula is the key tool to study the Riemann surfaces. We state a weaker version below.

Theorem 5 *(Riemann's inequality, a weaker version of Riemann–Roch) Let S be a Riemann surface of genus g and $x_1, \ldots, x_d \in S$. For any positive integers $m_1, \ldots, m_d$, the space $L(\sum_{i=1}^d m_i x_i)$ of meromorphic functions, which have a pole at x_i with order $\leq m_i$ and holomorphic elsewhere, is a $\mathbb{C}$-vector space of dimension $\geq (\sum_{i=1}^d m_i) + 1 - g$.*

[2]This is a highly nontrivial fact. It is a key step to show that every Riemann surface is an algebraic curve.

For example, there is a nonconstant meromorphic function if we allow poles with $\sum_{i=1}^d m_i > g$. Riemann–Roch's formula is:

$$H^0(S, \mathcal{O}(D)) - H^1(S, \mathcal{O}(D)) = d + 1 - g,$$

where $D = \sum_{i=1}^d m_i x_i$ and $H^0(S, \mathcal{O}(D)) = L(\sum_{i=1}^d m_i x_i)$.

Theorem 6 *Let f be a nonzero meromorphic function. Then the number of poles (counted with multiplicities) equals the number of zeros.*

Proof. Let $\omega = d(\log f) = \dfrac{df}{f}$. If p is a zero (resp. a pole) of f of order k, then $\mathrm{Res}_p \omega = k$ (resp. $-k$). The result follows from Theorem 2. □

I want to emphasize that unlike the objects in differential geometry, the existence of certain function/form on S is a big issue. If you find some object on S, then it will give you some clue to the symmetry (or a property) of S.

Theorem 7 *Any two meromorphic functions $f, g \in \mathcal{M}(S)$ are algebraically related, that is, there exists a bivariate polynomial P such that $P(f, g) \equiv 0$.*

Does this look plausible to you? (For example, how do you see that any two polynomials $f(z)$ and $g(z)$ are algebraically related?[3]) As you see below, this can be shown by using the fact that every meromorphic function on $\mathbb{CP}^1$ is rational.

Proof. Let $d := \displaystyle\sum_{p\,:\,poles} \mathrm{ord}_p f$. Then f can be thought as a $d : 1$ branched covering $S \to \mathbb{CP}^1$. (This is another nice viewpoint for Riemann surfaces.)

For a small open set $U \subset S$ with local chart x, Let $z_1, \ldots, z_d$ be the local charts of $f^{-1}(U)$ such that $f^{-1}(x) = \{z_1, \ldots, z_d\}$. Let

$$\phi(x, y) \;=\; (y - g(z_1))(y - g(z_2)) \cdots (y - g(z_d)).$$

[3]This can be shown by linear algebra. Suppose $\deg f = m$ and $\deg g = n$. For large N, the vector space generated by $f^k g^\ell$'s for $k \le \frac{N}{2m}$ and $\ell \le \frac{N}{2n}$ has dimension $\le \frac{N^2}{4mn}$. On the other hand, this space is contained in $\langle 1, z, z^2, \ldots, z^N \rangle$ which has dimension $N + 1$. As N gets large enough, $\frac{N^2}{4mn} > N + 1$ and so there should be some polynomial relation between f and g.

This is a polynomial in y, whose coefficients $s_i(x), 1 \le i \le d$, are meromorphic functions : $U \to \mathbb{CP}^1$. These function can be extended to the whole $\mathbb{CP}^1$, including the branch points. Therefore $s_i \in \mathcal{M}(\mathbb{CP}^1)$ and they are rational functions in x. This confirms that $\phi(x, y)$ is a rational function in x and y and $\phi(f(z), g(z)) \equiv 0$ on S. □

As an example, recall the Weierstrass $\mathcal{P}$-function on T_τ:

$$\mathcal{P}(z) := \frac{1}{z^2} + \sum_{w \,\in\, \mathbb{Z}+\tau\mathbb{Z}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

Its derivative is given by

$$\mathcal{P}'(z) = -2 \sum_{w \,\in\, \mathbb{Z}+\tau\mathbb{Z}} (z-w)^{-3}.$$

Then $(\mathcal{P}'(z))^2 = 4\mathcal{P}(z)^3 - g_2\mathcal{P}(z) - g_3$ for some $g_2, g_3 \in \mathbb{C}$ which are computed from the lattice $\mathbb{Z} \oplus \mathbb{Z}\tau$. This gives the "algebraic equation" of T_τ given by

$$E = \{(x, y) \in \mathbb{C}^2 \ : \ y^2 = 4x^3 - g_2 x - g_3\}.$$

More details: Define

$$\Phi : T_\tau \to \overline{E} = E \cup \{\infty\} \subset \mathbb{CP}^2$$

by $\Phi(z) = (\mathcal{P}(z) : \mathcal{P}'(z) : 1)$. It can be checked that this is an embedding of T_τ into $\mathbb{CP}^2$ whose image is an algebraic curve (zero set of a plane cubic curve). The coefficients g_2 and g_3 (more precisely, the j-invariant $= g_2^3 - 27g_3^2$) provide the moduli of Riemann surfaces of genus 1 (= elliptic curves).