Lecture notes : 16th KIAS Geometry Winter School

Hyun Kyu Kim (email: hyunkyukim@ewha.ac.kr)

title: Basics on Teichmüller spaces of Riemann surfaces
date: 2021.Jan.19

---

references (공부하기 좋은 책들):

Farb-Margalit, *A primer on mapping class groups*
Hubbard, *Teichmüller theory*

목표 (to understand) :

Riemann (1857) said : "a curve of genus $g \geq 2$ depends on $3g-3$ moduli"

(curve는 Riemann surface를 의미. moduli는 (complex) parameter)

Let S = topological 2-manifold

Today : S compact of genus g, oriented.
We fix any smooth structure on S.

Def. A **holomorphic structure** (or **complex-analytic structure** or **complex structure**) on S is a maximal holomorphic atlas Σ on S.

(즉 $\Sigma = \{(U_i, \varphi_i)\}_{i \in I}$, where $\{U_i\}_{i\in I}$ is an open cover of S, $\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}$ diffeomorphism to an open subset of $\mathbb{C}$, and each transition map $\varphi_j \circ \varphi_i^{-1}$ is (bi)holomorphic)

$\to (S, \Sigma)$ is called a **Riemann surface** of topological type S.

Def. Riemann's classical **moduli space**

$$\mathcal{M}(S) := \{\text{holomorphic structures } \Sigma \text{ on } S\}/\mathrm{Diff}^+(S)$$

where the quotient means quotient by the action of pullback by orientation-preserving diffeomorphisms $S \to S$.

(cf. $\{$Riemann surfaces of topological type $S\}$/isomorphism)
(cf. $\mathcal{M}_g$)

want :
- parametrize all points of $\mathcal{M}(S)$ by parameters?
- structures on $\mathcal{M}(S)$? topologize? manifold?

왜 하나?

- 결과론 (해보니 좋더라)
- to quantize (하나씩만 다룰 수는 없고, 모아놓은 공간(위의 함수)을 다뤄야 된다)

어쨌든, $\mathcal{M}(S)$를 공부하려면, better idea to study:

Def. **Teichmüller space**

$$\mathcal{T}(S) = \{\text{holomorphic structures } \Sigma \text{ on } S\}/\mathrm{Diff}(S)_0$$

where the quotient is quotient by the action of pullback by diffeomorphisms $S \to S$ isotopic to identity.

Note: people write $\mathcal{T}(S) = \{\text{holomorphic structures } \Sigma \text{ on } S\}/\text{isotopy}$

We will see partially:

Thm (Fenchel-Nielsen $\sim$ 1948) *If $g \geq 2$, $\mathcal{T}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$.*

(we will see how the number $6g - 6$ comes up)

We will not see:

Thm. $\exists$(Weil-Petersson) *Kähler metric on $\mathcal{T}(S)$.*

Relationship with $\mathcal{M}(S)$?

Def. **Mapping class group**

$$\mathrm{MCG}(S) := \mathrm{Diff}^+(S)/\mathrm{Diff}(S)_0$$

(discrete group of components of $\mathrm{Diff}^+(S)$)

Note: $\mathrm{MCG}(S)$ acts 'nicely' on $\mathcal{T}(S)$, and $\mathcal{T}(S)/\mathrm{MCG}(S)$ is homeomorphic to $\mathcal{M}(S)$
(so $\mathcal{T}(S)$ is a universal cover of $\mathcal{M}(S)$)

Let's study $\mathcal{T}(S)$. People say

$$\mathcal{T}(S) \overset{1:1}{\longleftrightarrow} \{\text{hyperbolic metrics on } S\}/\text{isotopy}$$

(where a hyperbolic metric is a Riemannian metric with constant sectional curvature -1) holds due to the 'Uniformization Theorem'.

Thm (the **Uniformization Theorem**; Koebe, Poincaré, 1907) *A simply-connected Riemann surface is isomorphic (i.e. biholomorphic) to exactly one of $\mathbb{C}$, $\mathbb{P}^1$, $\mathbb{H}^2$.*

Here $\mathbb{P}^1$ is the Riemann sphere (can be viewed as $\mathbb{C} \cup \{\infty\}$, and is homeomorphic to S^2), and $\mathbb{H}^2 = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ is the upper half-plane. Note $\mathbb{H}^2$ is naturally a Riemann surface, whose holomorphic structure naturally inherits from that of $\mathbb{C}$ because $\mathbb{H}^2$ is an open subset of $\mathbb{C}$. Note that both $\mathbb{C}$ and $\mathbb{H}^2$ can be viewed as open submanifolds of $\mathbb{P}^1$.

Let's use the Uniformization Theorem.

Start from a Riemann surface (S, Σ) of topological type S (i.e. $[\Sigma] \in \mathcal{T}(S)$).

Let $p : \widetilde{S} \to S$ be a universal covering.

<u>Claim</u>. $\exists!$ *holomorphic structure* $\widetilde{\Sigma}$ *on* $\widetilde{S}$ *s.t.* $p : (\widetilde{S}, \widetilde{\Sigma}) \to (S, \Sigma)$ *is a local biholomorphism.*

Then $(\widetilde{S}, \widetilde{\Sigma})$ is a simply-connected Riemann surface, so by the Uniformization Theorem, $\exists$isomorphism

$$\Phi : (\widetilde{S}, \widetilde{\Sigma}) \to X$$

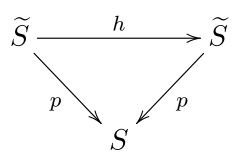
of Riemann surfaces, where

$$X \in \{\mathbb{C}, \mathbb{P}^1, \mathbb{H}^2\}.$$

How do we get the original (S, Σ) back? Quotient by the **deck transformation group**

$$\mathrm{Aut}(p) := \{h : \widetilde{S} \to \widetilde{S} \text{ homeomorphism s.t. } p = ph\}$$

(i.e. self-homeomorphisms h of $\widetilde{S}$ that preserve the covering p, i.e. h making the diagram



to commute)

Facts from 대수위상:
(1) $\mathrm{Aut}(p) \cong \pi_1(S)$
(2) $\mathrm{Aut}(p)$ acts properly discontinuously (and freely) on $\widetilde{S}$
(3) $\widetilde{S}/\mathrm{Aut}(p)$ is homeomorphic to S

Note : Consider the automorphism group of the Riemann surface $(\widetilde{S}, \widetilde{\Sigma})$:

$$\mathrm{Aut}(\widetilde{S}, \widetilde{\Sigma}) = \{h : \widetilde{S} \to \widetilde{S} \text{ homeomorphisms preserving } \widetilde{\Sigma}\}$$

(you may replace 'homeomorphisms' by 'diffeomorphisms'). Then the subgroups $\mathrm{Aut}(p)$ and $\mathrm{Aut}(\widetilde{S}, \widetilde{\Sigma})$ of $\mathrm{Homeo}(\widetilde{S})$ satisfy

$$\mathrm{Aut}(p) \leq \mathrm{Aut}(\widetilde{S}, \widetilde{\Sigma})$$

(by construction of $\widetilde{\Sigma}$)

Note : From the isomorphism $\Phi : (\widetilde{S}, \widetilde{\Sigma}) \to X$ of Riemann surfaces, we get the isomorphism

$$\mathrm{Ad}_\Phi : \mathrm{Aut}(\widetilde{S}, \widetilde{\Sigma}) \to \mathrm{Aut}(X), \quad h \mapsto \Phi \circ h \circ \Phi^{-1}$$

of the isomorphism groups.
(consider the diagram

$$\begin{array}{ccc} (\widetilde{S}, \widetilde{\Sigma}) & \xrightarrow{\Phi} & X \\ {\scriptstyle h}\downarrow & & \vdots \\ (\widetilde{S}, \widetilde{\Sigma}) & \xrightarrow{\Phi} & X \end{array}$$

)

<u>summary</u> (for a fixed $x \in S$)

$$\pi_1(S, x) \cong \mathrm{Aut}(p) \le \mathrm{Aut}(\widetilde{S}, \widetilde{\Sigma}) \underset{\cong}{\xrightarrow{\mathrm{Ad}_\Phi}} \mathrm{Aut}(X)$$

<u>Thm</u>.
(1) The resulting homomorphism

$$\rho = \rho_\Sigma : \pi_1(S, x) \to \mathrm{Aut}(X),$$

called the **monodromy representation** *(or* **holonomy representation***) of* Σ*, is well-defined up to conjugation by an element of* $\mathrm{Aut}(X)$*.*

(2) $\rho_\Sigma = \rho_{\Sigma'} \Leftrightarrow \Sigma, \Sigma'$ *represent a same point of* $\mathcal{T}(\Sigma)$*.*

(3) ρ *is faithful (i.e. injective) and (the image is) discrete.*

Question: $\mathrm{Aut}(X)$는 어떻게 생겼나?

<u>Thm</u>.
(1) $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm \mathrm{Id}\}$*, for each* $\sigma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$ *(i.e.* $a, b, c, d \in \mathbb{C}$*,* $ad - bc = 1$*), the associated automorphism* $f_\sigma \in \mathrm{Aut}(\mathbb{P}^1)$ *of* $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ *is* $f_\sigma(z) = \frac{az+b}{cz+d}$*)*

(2) $\mathrm{Aut}(\mathbb{C}) \cong \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C}) \,|\, a \neq 0\}$ *(so* $f : \mathbb{C} \to \mathbb{C}$ *given by* $f(z) = az + b$*)*

(3) $\mathrm{Aut}(\mathbb{H}^2) \cong \mathrm{PSL}(2, \mathbb{R})$

<u>Thm</u>. *Suppose that a Riemann surface* Y *is not isomorphic to* $\mathbb{P}^1$*,* $\mathbb{C}$*,* $\mathbb{C} \setminus \{a\}$*, or a Riemann surface of genus* 1 *(i.e. homeomorphic to the torus* $S^1 \times S^1$*). Then* Y *has* $\mathbb{H}^2$ *as its universal covering Riemann surface.*

<u>Thm</u>. *For* $g \ge 2$*,*

$$\mathcal{T}(S) \to \mathrm{Hom}^{df}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})$$

sending $[\Sigma]$ *to* $[\rho_\Sigma]$ *is a bijection.*

Here, the right-hand-side is the set of all group homomorphisms $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ defined up to conjugation by an element of $\mathrm{PSL}(2,\mathbb{R})$ such that ρ is faithful and (the image of ρ is) discrete.

Can we see $\dim \mathcal{T}(S)$? ($\mathcal{T}(S)$ locally homeomorphic to $\mathbb{R}^N$, for which N?)

$\pi_1(S)$ has a presentation by $2g$ generators and 1 relation. Note that $\mathrm{PSL}(2,\mathbb{R})$ is a real 3-manifold. So, at least locally,

$$\dim \mathcal{T}(S) = (2g-1)\cdot 3 - 3,$$

where -3 comes from the quotient action by the 3-manifold Lie group $\mathrm{PSL}(2,\mathbb{R})$. So

$$\dim \mathcal{T}(S) = 6g - 6.$$

What we didn't see today : the hyperbolic metric description of $\mathcal{T}(S)$. We will see next time a little bit (how a Riemann surface structure Σ on S induce a hyperbolic metric on S)