Lecture notes : 16th KIAS Geometry Winter School

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title: Teichmüller spaces as cluster varieties
date: 2021.Jan.25

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1 Continuing from last time : compact surfaces

Let S be a compact oriented surface. The **Teichmüller space** is

$$\mathcal{T}(S) := \{\text{holomorphic structures } \Sigma \text{ on } S\}/\text{isotopy}$$

which is in bijection with

$$\mathrm{Hom}^{df}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}),$$

the set of all group homomorphisms $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, defined up to conjugation by an element of $\mathrm{PSL}(2,\mathbb{R})$ (i.e. two homomorphisms $\rho, \rho' : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$ are viewed as equivalent if $\exists \sigma \in \mathrm{PSL}(2,\mathbb{R})$, $\forall [\gamma] \in \pi_1(S)$, $\rho'([\gamma]) = \sigma\, \rho([\gamma])\, \sigma^{-1}$; to be more precise, pick a point x of S and replace $\pi_1(S)$ by $\pi_1(S,x)$), such that ρ has discrete image in $\mathrm{PSL}(2,\mathbb{R})$ and ρ is faithful (i.e. injective).

For a point $[\Sigma]$ of $\mathcal{T}(S)$ represented by the holomorphic structure Σ on S, let $[\rho] = [\rho_\Sigma]$ be the corresponding point of $\mathrm{Hom}^{df}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ represented by the monodromy representation homomorphism

$$\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R}).$$

Let

$$\Gamma := \rho(\pi_1(S)) \le \mathrm{PSL}(2,\mathbb{R})$$

be the corresponding **Fuchsian group** (this just means a discrete subgroup of $\mathrm{PSL}(2,\mathbb{R})$). We have the commutative diagram of maps

$$\begin{array}{ccc} (\widetilde{S}, \widetilde{\Sigma}) & \xrightarrow{\ \Phi\ } & (\mathbb{H}^2, \Sigma_{\mathbb{H}^2}) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \text{quotient}} \\ (S, \Sigma) & \overset{\phi}{\dashrightarrow} & \mathbb{H}^2/\Gamma \end{array}$$

for a unique map $\phi : (S,\Sigma) \to \mathbb{H}^2/\Gamma$, which is an isomorphism of Riemann surfaces, hence in particular a diffeomorphism from S to $\mathbb{H}^2/\Gamma$.

Note that $\mathbb{H}^2 = \{x + \mathrm{i}y \in \mathbb{C} \mid y > 0\} = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ has a standard *hyperbolic metric* $g_{\mathbb{H}^2} = \frac{dx^2+dy^2}{y^2}$ (a hyperbolic metric means a Riemannian metric with sectional curvature$\equiv 1$; since we are dealing only with surfaces here, one can replace the 'sectional curvature' by the 'Gaussian curvature'). The automorphism group of this Riemannian manifold $(\mathbb{H}^2, g_{\mathbb{H}^2})$ is:

Thm. *The orientation-preserving isometry group*

$$\mathrm{Isom}^+(\mathbb{H}^2, g_{\mathbb{H}^2}) := \{h : \mathbb{H}^2 \to \mathbb{H}^2 \text{ orientation-preserving diffeomorphisms preserving } g_{\mathbb{H}^2}\}$$

(here, h preserving $g_{\mathbb{H}^2}$ means $h^* g_{\mathbb{H}^2} = g_{\mathbb{H}^2}$) coincides with the group $\mathrm{PSL}(2, \mathbb{R}) \le \mathrm{Homeo}(\mathbb{H}^2)$ *of* $\mathrm{PSL}(2, \mathbb{R})$ *fractional linear transformations.*

Hence

$$\mathrm{Isom}^+(\mathbb{H}^2, g_{\mathbb{H}^2}) = \mathrm{PSL}(2, \mathbb{R}) = \mathrm{Aut}(\mathbb{H}^2, \Sigma_{\mathbb{H}^2}).$$

(preserving lengths and angles, i.e. being an isometry
$\leftrightarrow$preserving angles, i.e. being conformal
$\leftrightarrow$preserving complex structures, i.e. being holomorphic)

So $\Gamma \le \mathrm{PSL}(2, \mathbb{R})$ is a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2, g_{\mathbb{H}^2})$, hence the quotient $\mathbb{H}^2/\Gamma$ inherits the hyperbolic metric. We can then pullback this hyperbolic metric along the diffeomorphism $\phi : S \to \mathbb{H}^2/\Gamma$, to obtain a hyperbolic metric $g_{[\Sigma]}$ on S.

Thm. *This hyperbolic metric $g_{[\Sigma]}$ on S is well-defined by $[\Sigma] \in \mathcal{T}(S)$, up to isotopy, and this assignment $[\Sigma] \mapsto g_{[\Sigma]}$ yields a bijection*

$$\mathcal{T}(S) \to \{\textit{hyperbolic metrics on } S\}/\textit{isotopy}$$

Some authors just define the Teichmüller space this way, i.e. as the space of all isotopy classes of hyperbolic metrics on S.

2 Teichmüller spaces for non-compact surfaces

This time, let S be a *non-compact* oriented topological 2-manifold. We assume S is of *finite type*, i.e. S is obtained from a compact oriented surface of genus g by removing n points, where the removed points are called the **punctures**. We require

$$n \ge 1, \qquad \chi(S) = 2 - 2g - n < 0.$$

Pick any smooth structure on S, so we may view S as a smooth oriented surface.

Def. *The* **Teichmüller space** *of S is defined as*

$$\mathcal{T}(S) = \{\textit{complete hyperbolic metrics } g \textit{ on } S\}/\textit{isotopy}$$

Here a hyperbolic metric g on S being *complete* means that, when S is viewed as a metric space with the metric defined from the lengths of paths with respect to g (distance between two points of S being the length of the shortest piecewise smooth path), S is a complex metric space. And modding out by isotopy means modding out by pullbacks by self-diffeomorphisms of S isotopic to the identify diffeomorphism.

It turns out that the above version $\mathcal{T}(S)$ is not so nice for our purpose.

For other versions to consider, we need to investigate the boundary behaviors of the hyperbolic metrics g on S, i.e. the asymptotic behavior near the punctures. It is convenient to describe such a behavior in terms of monodromy. First, we need:

Thm. *The assignment to each hyperbolic metric on S its monodromy representation yields the bijection*

$$\mathcal{T}(S) \leftrightarrow \mathrm{Hom}^{df;S}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R}),$$

where the right hand side is the set of all group homomorphisms $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$, defined up to conjugation, such that ρ has discrete image, ρ is faithful, and ρ is of 'type S', i.e. $\mathbb{H}^2/\rho(\pi_1(S))$ is homeomorphic to S.

From now on, a point of $\mathcal{T}(S)$ may be denoted by $[g]$ (isotopy class of a hyperbolic metric).

Def. *A non-identity element A of* $\mathrm{PSL}(2,\mathbb{R})$ *is called*

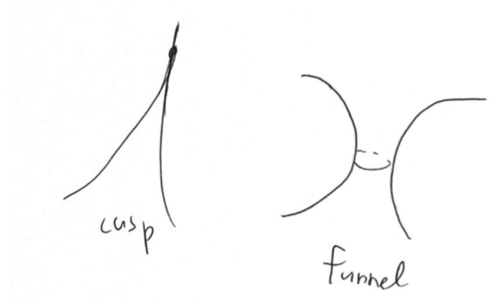
- **parabolic** *if* $|\mathrm{tr}(A)| = 2$;
- **hyperbolic** *if* $|\mathrm{tr}(A)| > 2$;
- **elliptic** *if* $|\mathrm{tr}A| < 2$

(each is similar to $\pm\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ $(a \neq 0)$, $\pm\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$ $(\lambda > 0)$, $\pm\left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ $(\theta \in \mathbb{R})$)

Note. *For $[\rho] \in \mathcal{T}(S)$, the value $\rho([\gamma])$ is never elliptic.*

Def. *Let p be a puncture of S. (so p is not really a point of S) Let γ_p be a small simple loop in S surrounding p. Let $[g] \in \mathcal{T}(S)$, and let $[\rho]$ be the corresponding conjugacy class of monodromy representation homomorphisms $\rho : \pi_1(S) \to \mathrm{PSL}(2,\mathbb{R})$.*

- *p is a* **cusp** *with respect to $[g]$ if $\rho([\gamma_p])$ is parabolic;*
- *p is a* **funnel** *with respect to $[g]$ if $\rho([\gamma_p])$ is hyperbolic.*



(cusp와 funnel 참조: 제 15회 고등과학원 기하학 겨울학교, 김영주 교수 강연 "쌍곡기하 산책")

Def. *Define the* **cusped (or, punctured) Teichmüller space** *as*

$$\mathcal{T}^u(S) := \{[g] \in \mathcal{T}(S) \,|\, \textit{all punctures of } S \textit{ are cusps with respect to } [\rho]\}$$

and the **enhanced (or, holed) Teichmüller space** *as*

$$\mathcal{T}^+(S) := \left\{ [g, O] \;\middle|\; \begin{array}{l} [g] \in \mathcal{T}(S),\, O \textit{ is a choice of orientation} \in \{+,-\} \\ \textit{per each funnel of } S \textit{ with respect to } [g] \end{array} \right\}$$

(I think u stands for 'unipotent')

The data O can be thought of as follows. For a funnel puncture p of S, there is a unique geodesic in the homotopy class of the loop γ_p around p. The data O is a choice of an orientation of this geodesic, per each funnel puncture p.

FACT. *The enhanced Teichmüller space $\mathcal{T}^+(S)$ is a smooth manifold diffeomorphic to*

$$\mathcal{T}^+(S) \approx \mathbb{R}^{6g-6+3n},$$

and is equipped with a Poisson structure called the Weil-Petersson Poisson structure. The cusped Teichmüller space $\mathcal{T}^u(S)$, viewed as a subspace of $\mathcal{T}^+(S)$, is a symplectic leaf with respect to this Poisson structure. Moreover, $\mathcal{T}^u(S)$ is diffeomorphic to

$$\mathcal{T}^u(S) \approx \mathbb{R}^{6g-6+2n},$$

possesses a Kähler structure compatible with the Weil-Petersson symplectic form, and can be viewed as a universal cover of Riemann's classical moduli space $\mathcal{M}_{g,n}$.

One of the goals of today's lecture is to partially understand the above FACT.

To start with, how do we investigate $\mathcal{T}^+(S)$? Say we do not know anything about the above FACT yet. We should first topologize the set $\mathcal{T}^+(S)$, and try to show that it is locally or globally homeomorphic to some $\mathbb{R}^N$. That is, we should find some charts, and maybe check if the charts we found are smoothly compatible with each other. One way to topologize the set $\mathcal{T}^+(S)$ is to use the model $\mathrm{Hom}^{df;S}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$. First, since $\pi_1(S)$ is a free group (why?), the set $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ can be identified with the product of bunch of $\mathrm{PSL}(2,\mathbb{R})$. As each $\mathrm{PSL}(2,\mathbb{R})$ is naturally a topological space, one can topologize $\mathrm{Hom}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ as $(\mathrm{PSL}(2,\mathbb{R}))^M$ for some M. Then $\mathrm{Hom}^{df;S}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))$ can be given the subspace topology, and finally $\mathrm{Hom}^{df;S}(\pi_1(S), \mathrm{PSL}(2,\mathbb{R}))/\mathrm{PSL}(2,\mathbb{R})$ the quotient topology. But we will not do it this way.

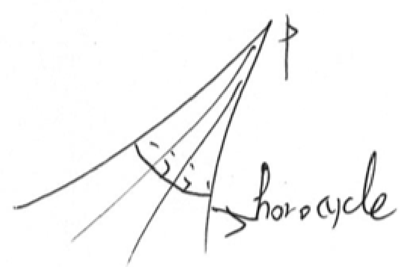
Instead, just viewing $\mathcal{T}^+(S)$ as a set, we will start finding charts for this set right away, without giving it a topology first. Then you can use '(smooth) manifold chart lemma' in the end (see Lee's book *Introduction to Smooth Manifolds*, 2nd ed., for example).

In fact, due to lack of time, we choose not to do this task for $\mathcal{T}^+(S)$ today, but only for the following yet another version of a Teichmüller space.

Def. *Define the* **decorated Teichmüller space** *as*

$$\mathcal{T}^d(S) := \{[g,h] \mid [g] \in \mathcal{T}^u(S),\ h \text{ is a choice of a horocycle at each puncture}\}$$

Here, a **horocycle** at a (cusp) puncture p with respect to a hyperbolic metric g on S means a (smooth) curve in S that is perpendicular to all geodesics with respect to g emanating from p. It can be viewed as a circle centered at ∞.



So we have the following diagram of maps between different version of Teichmüller spaces

$$\begin{array}{ccc} \mathcal{T}^d(S) & & \mathcal{T}^+(S) \\ \downarrow & & \downarrow \\ \mathcal{T}^u(S) & \hookrightarrow & \mathcal{T}(S) \end{array}$$

where the vertical maps are maps forgetting decoration and orientation, respectively. The left vertical map $\mathcal{T}^d(S) \to \mathcal{T}^u(S)$ is a fiber bundle with fiber diffeomorphic to $\mathbb{R}^n$, and the right vertical map $\mathcal{T}^+(S) \to \mathcal{T}(S)$ is a 2^n-to-1 branched covering. Note that the bottom right $\mathcal{T}(S)$ is the naive definition of the Teichmüller space, while the bottom left $\mathcal{T}^u(S)$ is the most 'nice' version, which had been classically studied (called $\mathcal{T}_{g,n}$, which is a universal cover of $\mathcal{M}_{g,n}$). One reason why people considered the upper versions $\mathcal{T}^d(S)$ and $\mathcal{T}^+(S)$ (in 1980's and on) is to obtain nice sets of coordinate systems, suitable for the quantization problem. However, even without the quantization, these special coordinate systems turned out to be providing prototypical examples of more general phenomenon, namely those of *cluster $\mathcal{A}$-varieties* (or cluster K_2-varieties) and *cluster $\mathscr{X}$-varieties* (or cluster Poisson varieties) which arose in the 21st century.

3 Coordinate system for the decorated Teichmüller space $\mathcal{T}^u(S)$

So, let us study coordinate systems for the various versions of Teichmüller spaces.

The original plan (for the last lecture) was to also review the Fenchel-Nielsen coordinate systems for $\mathcal{T}(S)$ (and $\mathcal{T}^u(S)$), which is most classical (early 20c), and based on a choice of an extra data called a *pants (or trouser) decomposition* of S. But let's not do it.. These coordinate systems turned out to be not suitable for the task of quantization.

The new sets of coordinate systems which arose in 1980's require the choice of the following data :

Def. *An* **ideal arc** *in S is an unoriented simple path running between (not necessarily distinct) punctures. An isotopy of ideal arcs means an isotopy (i.e. homotopy) within the class of ideal arcs.*

An **ideal triangulation** *T of S is a collection of ideal arcs in S such that*

(1) *no arc of T bounds a disk;*

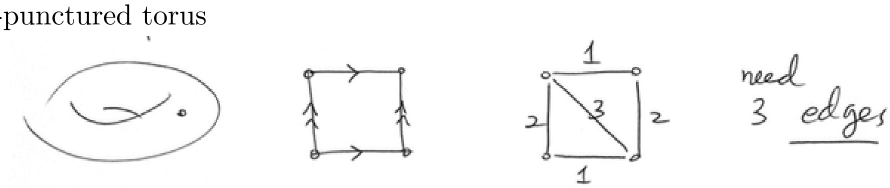
(2) *no two members are isotopic or intersect in S;*

(3) *the collection T is maximal among the collections satisfying (1)–(2).*

Note: T divides S into *ideal triangles*

Note: We often consider T only up to (simultaneous) isotopy.

e.g. once-punctured torus



e.g. 3-punctured sphere

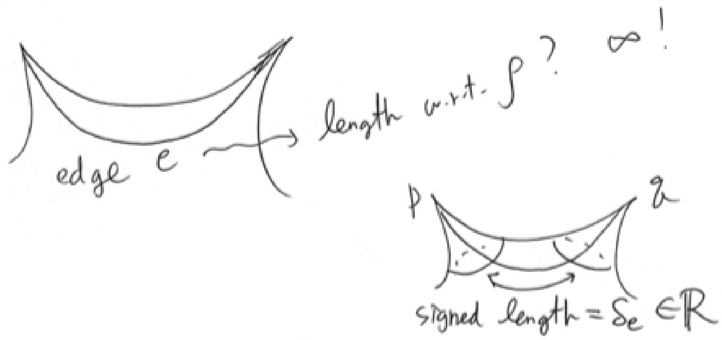
Note: T has $6g - 6 + 3n$ constituent arcs. (exercise)

Let's now describe a coordinate system for the decorated Teichmüller space $\mathcal{T}^d(S)$.

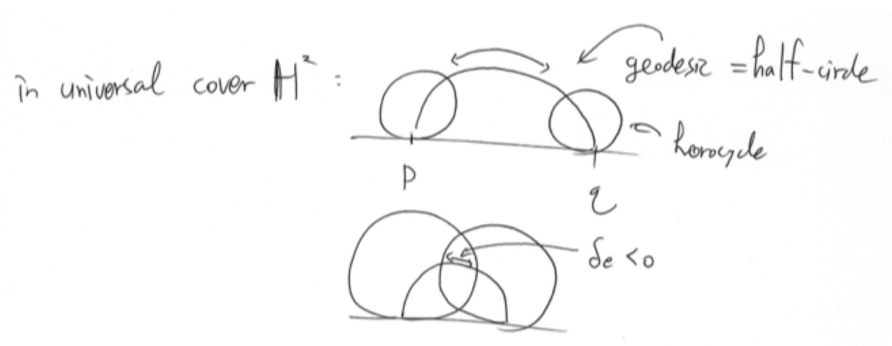
Choose one ideal triangulation T of S.

Given a point $[\rho, h] \in \mathcal{T}^d(S)$, how do we get coordinates, i.e. some set of real numbers representing this point?

Stretch all ideal arcs of T to unique geodesics ('shortest' paths) with respect to g, in their respective isotopy classes. (question: why possible? why unique? exercise!)



Then, if you try to measure the geodesic length of an arc e of T with respect to g, you get ∞. Truncate both ends of e by the horocycles at the endpoint punctures, and then measure the geodesic length δ_e of the remaining finite-length part of e. So, δ_e can be thought of as the distance between the horocycles. In fact, δ_e is defined as the *signed* distance between the horocycles. That is, if the horocycles are too big so that they truncate too much as in the figure below, we set δ_e to be minus the distance between the horocycles.



Def. *Penner's* **lambda length** *for* $[g, h] \in \mathcal{T}^d(S)$ *at arc* e *of* T *is*

$$\lambda_{e,T}([g, h]) := \exp(\delta_e/2) \in \mathbb{R}_{>0}.$$

Note that $\lambda_{e,T}([g, h])$ does not depend on T, so one can write it as $\lambda_e([g, h])$.

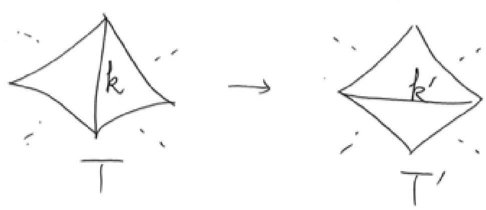
Thm. (Penner, 1980's) *For each ideal triangulation T, the map*

$$\begin{aligned}\mathcal{T}^d(S) &\to (\mathbb{R}_{>0})^T \\ [g,h] &\mapsto (\lambda_e([g,h]))_{e\in T}\end{aligned}$$

is a diffeomorphism.

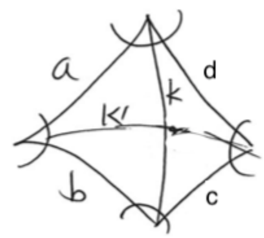
In fact, at this point, we can't say that this is a diffeomorphism. We can only understand this statement as saying that this map is a set bijection.

A crucial point is what happens if we choose a different ideal triangulation T'. Easiest way to choose a different T' is to replace exactly one arc k of T by another one; this process is called a **flip** at k:



Then, only one arc changes, so $\lambda_{e,T} = \lambda_{e,T'}$ for all arcs e other than k. What about the coordinate $\lambda_{k'} = \lambda_{k',T'}$ for the new arc k'?

Thm. ("Ptolemy relation") *If a,b,c,d are sides of an ideal quadrilateral, and if k,k' are two ideal diagonals of this quadrilateral,*



then the lambda length coordinate functions satisfy

$$\lambda_k\lambda_{k'} = \lambda_a\lambda_c + \lambda_b\lambda_d$$

Note: this is an example of an "exchange relation", which looks like

$$\text{new}\cdot\text{old} = \text{product} + \text{product}.$$

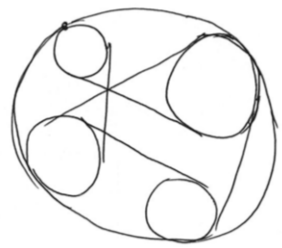
Another example of such a relation is the famous Plücker relation for Grassmannians.

Note: so

$$\lambda_{k'} = \frac{1}{\lambda_k}(\lambda_a\lambda_c + \lambda_b\lambda_d)$$

Note: This pretty coordinate change formula justifies the strange factor 1/2 in the exponent of the definition of the lambda length $\lambda_e = e^{\delta_e/2}$.

Note: The actual "Ptolemy's Theorem" is for the Euclidean lengths of the Euclidean quadrilateral inscribed in a circle. The above version of Ptolemy's theorem is an incarnation of 'Casey's Theorem' of 19c.



4 Cluster varieties

To touch upon the subject of cluster varieties, we now focus on the coordinate change formulas only, and not the 2d geometry.

Per each ideal triangulation T of S, consider the set of commuting formal variables $\lambda_1, \ldots, \lambda_N$ (where $N = |T| = 6g - 6 + 3n$) enumerated by the arcs of T, and consider the split algebraic torus

$$\mathcal{A}_T = (\mathbb{G}_m)^N = \mathrm{Spec}(\mathbb{Z}[\lambda_1^{\pm 1}, \ldots, \lambda_N^{\pm 1}]),$$

which is an affine scheme whose ring of regular functions is the N-variable Laurent polynomial ring. Such a data is called a **seed**.

For another ideal triangulation T', we would have another variables $\lambda'_1, \ldots, \lambda'_N$, and the corresponding affine scheme

$$\mathcal{A}_{T'} = (\mathbb{G}_m)^N = \mathrm{Spec}(\mathbb{Z}[(\lambda'_1)^{\pm 1}, \ldots, (\lambda'_N)^{\pm 1}]).$$

We glue the tori $(\mathbb{G}_m)^N$ and $(\mathbb{G}_m)^N$ for T and T' along the rational map which we denote by

$$\mu_{TT'} : \mathcal{A}_T \to \mathcal{A}_{T'}$$

which is defined by the coordinate change formulas for the lambda lengths. Namely, for example, if T and T' are related by the flip at edge k, then $\mu_{TT'}$ is defined so that its pullback is given by

$$\mu^*_{TT'}(\lambda'_e) = \begin{cases} \lambda_e & \text{if } e \neq k \\ \frac{1}{\lambda_k}(\lambda_a \lambda_c + \lambda_b \lambda_d) & \text{if } e = k, \end{cases}$$

where a, b, c, d are the arcs of T forming the ideal quadrilateral (in this counterclockwise orderr) having k as a diagonal.

Consider all possible ideal triangulations T of S, and glue all the tori $\mathcal{A}_T$ together by these maps $\mu_{TT'}$, called **mutations**. The resulting scheme is an example of the **cluster $\mathscr{A}$-variety**. To be a bit more precise, a cluster $\mathscr{A}$-variety is associated to a quiver (i.e. directed graph). In our case, each ideal triangulation T yields a quiver Q_T constructed as follows: each arc of T has a node of Q on it, and for each ideal triangle we draw the three arrows between the nodes at the sides of this triangle, forming a counterclockwise 3-cycle. The cluster $\mathscr{A}$-variety constructed above is the cluster $\mathscr{A}$-variety associated to the quiver Q_T.

Note: similar construction for the enhanced Teichmüller space $\mathcal{T}^+(S)$ with exponentiated (Thurston's) shear coordinate functions $X_{e,T} = e^{x_{e,T}}$ (defined per each arc e of T) yield the **cluster $\mathscr{X}$-variety** for the quiver Q_T.

Note: there is no subtraction involved in the coordinate change formula $\rightsquigarrow$ this allows us to evaluate the cluster variety at a *semi-field* (which is like a field without subtraction), e.g. $\mathbb{R}_{>0}$.

Note: we can now use tools from the theory of cluster algebras and cluster varieties, to study Teichmüller spaces. Especially for the quantization problem (with respect to the Weil-Petersson Poisson structure on $\mathcal{T}^+(S)$, which can be interpreted in terms of a generalized combinatorial description for cluster $\mathscr{X}$-varieties).

More details on cluster varieties and their quantization can be found in:

V.V. Fock and A.B. Goncharov, *Moduli spaces of local systems and higher Teichmüller theory*, Publ. Math. Inst. Hautes Études Sci. **103** (2006), 1–211. arXiv:math/0311149

V.V. Fock and A.B. Goncharov, *The quantum dilogarithm and representations of quantum cluster varieties*, Invent. Math. **175** (2) (2009), 223-286. arXiv:math/0702397