The following are the video link and slides for Eric Rowell’s Lecture 2.

https://livewarwickac-my.sharepoint.com/personal/u1972347_live_warwick_ac_uk/_layouts/15/onedrive.aspx?id=%2Fpersonal%2Fu1972347%5Flive%5Fwarwick%5Fac%5Fuk%2FDocuments%2FAGQFT%2Frowell%2Dkias%2D2%2Emp4&parent=%2Fpersonal%2Fu1972347%5Flive%5Fwarwick%5Fac%5Fuk%2FDocuments%2FAGQFT&originalPath=aHR0cHM6Ly9saXZld2Fyd2lja2FjLW15LnNoYXJlcG9pbnQuY29tLzp2Oi9nL3BlcnNvbmFsL3UxOTcyMzQ3X2xpdmVfd2Fyd2lja19hY191ay9FWnhFSnNvRmxsQkpuZXN4b0FUUkZZVUJwQ0ktNFRodGtscDNicGp0UnRHNW9nP3J0aW11PU5Gbm5odUZYMlVn

Computational Power 💪

- How does TQC compare with QCM?
- What problems can a TQC handle?

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1. Can efficiently simulate TQC on (Universal) QCM
2. Conversely!
3. Classically hard problems on TQC

Defn: Class of decision problems efficiently solvable on a QC: BQP

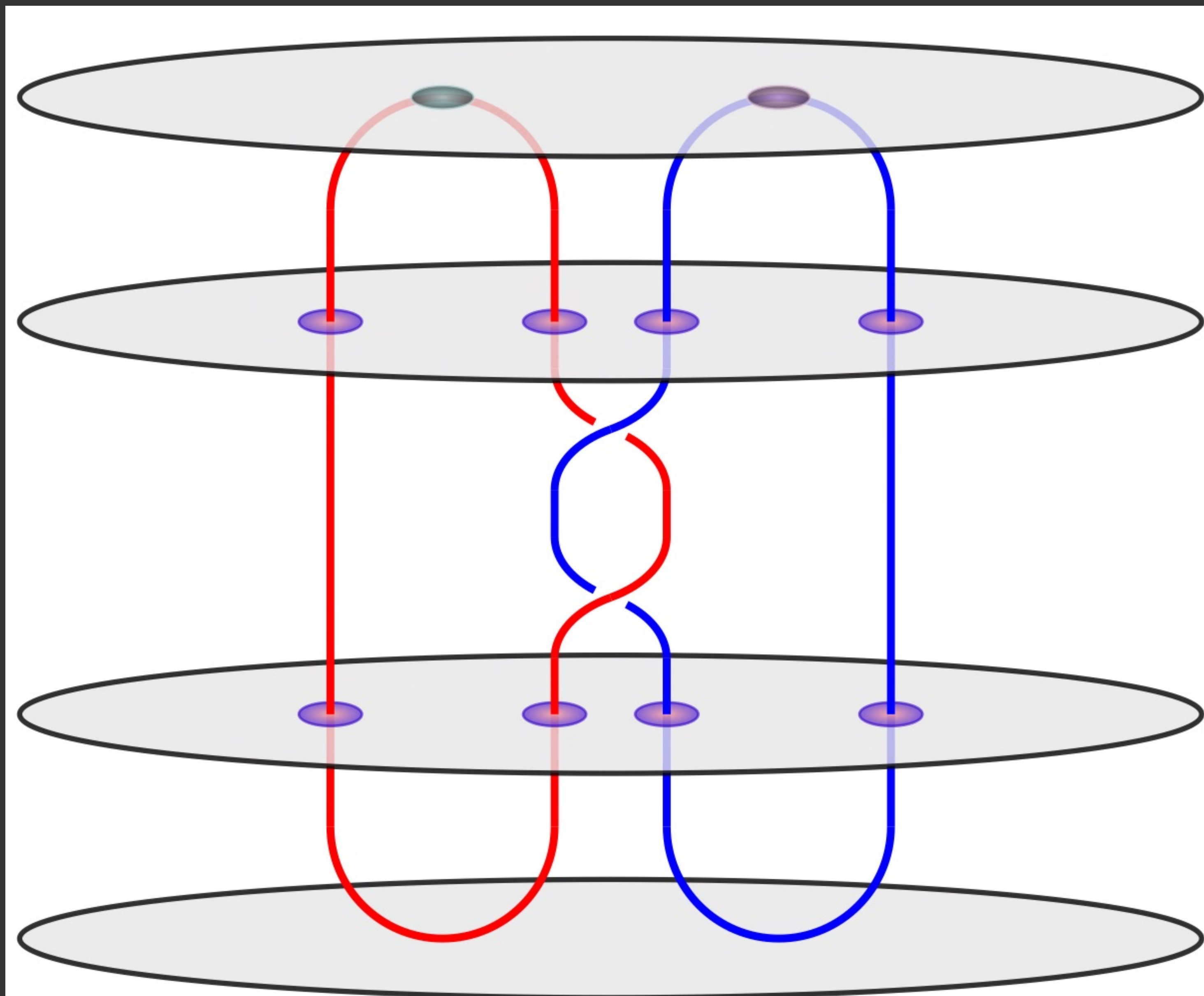
Bounded error, Quantum resource, Poly'l time.

Typical topological Quantum process

measure

braid (gates)

create

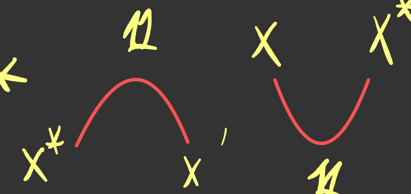


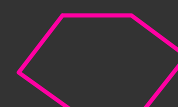
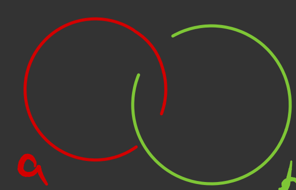
t

A pocket Dictionary (details later)

UMC	*Anyonic system*
simple object	anyon
label	anyon type or topological charge
tensor product	fusion
fusion rules	fusion rules
triangular space V_{ab}^{c} or V_{c}^{ab}	fusion/splitting space
dual	antiparticle
birth/death	creation/annihilation
mapping class group representations	anyon statistics
nonzero vector in $V(Y)$	ground state vector
unitary F-matrices	recoupling rules
twist $\theta_x = e^{2\pi i s_x}$	topological spin
morphism	physical process or operator
tangles	anyon trajectories
quantum invariants	topological amplitudes

Modular Category $\mathcal{C}$ (over $\mathbb{C}$)

- **Monoidal**: $(\otimes, \mathbb{1}, \alpha)$:  axioms
- **Semisimple**: $X \cong \bigoplus_i X_i^{\oplus m_i}$
- **Linear**: $\mathrm{Hom}(X,Y) \in \mathrm{Vec}_{\mathbb{C}}$
- **Rigid**: $X^* \otimes X \to \mathbb{1} \to X \otimes X^*$ 

- **Braided**: $X \otimes Y \overset{c}{\cong} Y \otimes X$  axioms
- **Finite Rank**: $|\mathrm{Irr}(\mathcal{C})| < \infty$
- **Spherical**: $\theta_X : X \cong X$  $\cong$ 
- **Non-degenerate**: $S_{ab} =$ 

invertible

Ribbon diagram graphical calculus

$X \cong X^{**}$

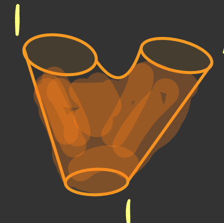
Trace on $\mathrm{End}(X)$

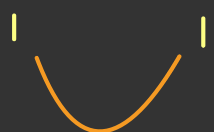
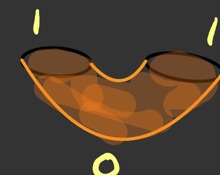
Fibonacci

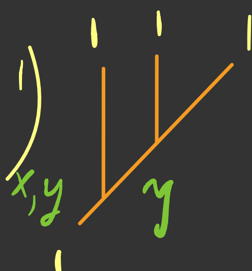
2 anyon types: $\{0, 1\}$ $\quad X_0 = \mathbb{1}$

$$N_{11}^{1} = 1 = N_{11}^{0} = N_{01}^{1} = N_{10}^{1} = N_{00}^{0} \qquad X_1 = \tau$$

$$\tau^* \cong \tau$$

$\mathbb{C}\{$  $\} = \mathrm{Hom}(\tau, \tau^{\otimes 2})$: 

$\mathbb{C}\{$  $\} = \mathrm{Hom}(\mathbb{1}, \tau^{\otimes 2})$ 

 $= \sum_y (F_1^{111})_{x,y}$ 

$$F_1^{111} = \begin{pmatrix} \varphi^{-1} & \varphi^{-\frac{1}{2}} \\ \varphi^{-\frac{1}{2}} & -\varphi \end{pmatrix} \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

$x, y \in \{0, 1\}$

cat. dim.

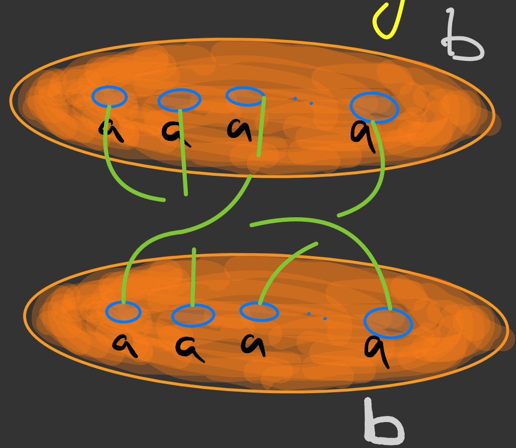
 $= \varphi$

 $= R_x^{11}$  $\quad R_1^{11} = e^{3\pi i/5}, \quad R_0^{11} = e^{-4\pi i/5}$

 $= e^{4\pi i/5}$ 

Set-up (Freedman-Kitaev-Wang)

○ $\mathcal{C}$ is a unitary MTC $X_b, X_a \in \mathcal{C}$.



$$\mathrm{Hom}(X_b, X_a^{\otimes n}) = \mathcal{S}_n \xleftarrow{\rho(\beta)} \mathrm{Hom}(X_b, X_a^{\otimes n}) = \mathcal{S}_n$$

○ Problem: Efficiently approximately simulate $\rho(\beta) \in U(\mathcal{S}_n)$ on universal QCM

○ All universal QCMs are (poly) equiv.: Qudits ~ Qubits

d-level $\xleftrightarrow{\text{poly equiv.}}$ 2-level. May choose any $d \geq 2$.

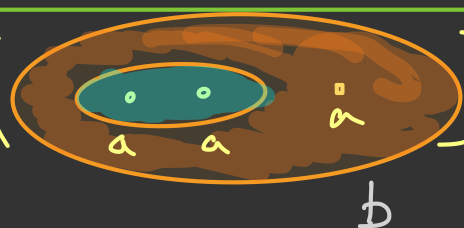
○ $\mathcal{G}$: universal Qudit Gate set GIVEN

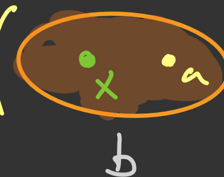
Let $Y := \bigoplus_{(a,b,c)\in\mathcal{L}^3} \mathcal{H}\Big($ 

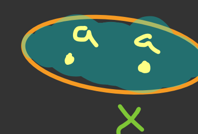
$\Big)$ $\qquad d = \dim(Y)$.

Ex: Fib: $d = \dim(Y) = 5$.

$\mathcal{S}_n \subseteq Y^{\otimes(n-1)}$: distribute $\otimes$ over $\oplus$.

Ex: $\mathcal{H}\Big($ 

$\Big) \cong \bigoplus_x \mathcal{H}\Big($ 

$\Big) \otimes \mathcal{H}\Big($ 

$\Big)$

$\mathcal{H}(\ldots) = \mathcal{S}_3$, $\quad \bigoplus_x \mathcal{H}(\ldots)\otimes\mathcal{H}(\ldots) \subseteq Y^{\otimes 2}$

$$Y^{\otimes(n-1)} = \mathcal{S}_n \oplus \mathcal{S}_n^{\perp}$$

$$\begin{tikzcd}
\mathcal{S}_n \arrow[r, hook] \arrow[d, "\rho(\beta)"'] & Y^{\otimes(n-1)} \arrow[d, "U"] \\
\mathcal{S}_n \arrow[r, hook] & Y^{\otimes(n-1)}
\end{tikzcd}$$

find

s.t.

$U \simeq \rho(\beta) \oplus \mathrm{Id}_{\mathcal{S}_n^{\perp}}$ i.e. $\| U - \rho(\beta) \oplus I \| < \varepsilon$

Possible by Universality & Kitaev-Solovay

More generally: simulate image of any $\varphi \in MCG(\Sigma_{g,n})$ via gluing & $\amalg$.

Thus: TQC is no stronger than QCM.

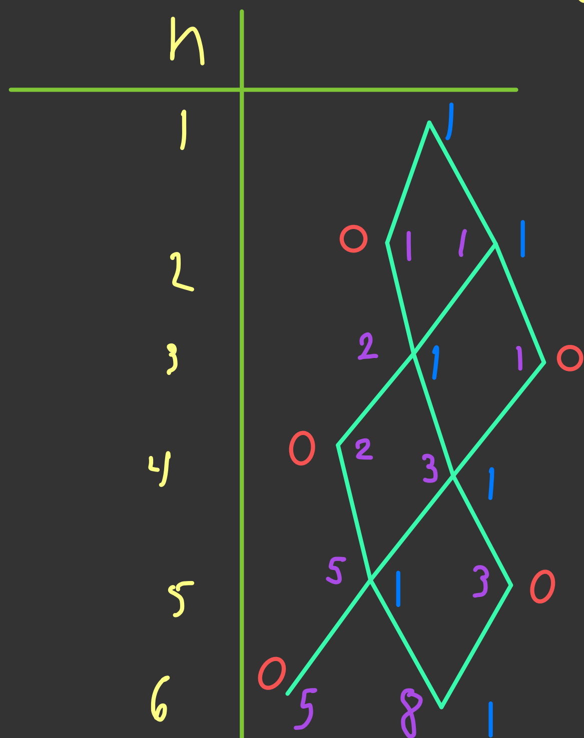
Other direction? Efficiently approximately simulate $U \in SU(\underbrace{\mathcal{H}_n}_{SU\,1}) = (\mathbb{C}^2)^{\otimes n}$ via braiding gates.

(Freedman–Larsen–Wang)

$X_0 = \mathbb{1}$

Focus on Fibonacci Theory: $X := X_1 = \tau$.

Bratteli diagram for $\mathrm{End}(X^{\otimes n})$



Irred. reps. $\begin{cases} V_n = \mathrm{Hom}(\mathbb{1}, \tau^{\otimes n}) \\ V_n' = \mathrm{Hom}(\tau, \tau^{\otimes n}) \end{cases}$

$\dim(V_n) = f_n \quad f_1 = 0, f_2 = 1$

$\dim(V_n') = f_{n+1}$

Fusion Rule:

$X^{\otimes 2} = X \oplus \mathbb{1}$

B_n acts irreducibly on V_n, V_n'.

$\rho(B_n) \subseteq U(V_n)$

$\rho'(B_n) \subseteq U(V_n')$

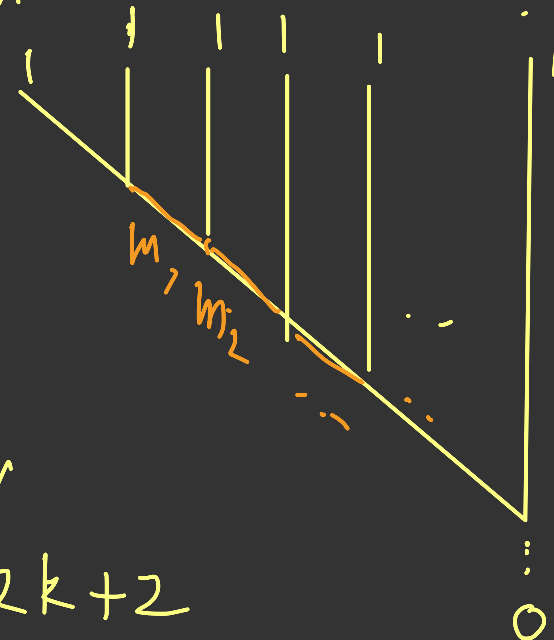
$(\rho, V_n') \cong (\rho, V_{n+1})$

$\underline{\text{Th}^m}$ $\overline{\rho(B_n)} \supseteq SU(V_n)$.

i.e. B_n image is dense in $SU(V_n)$.

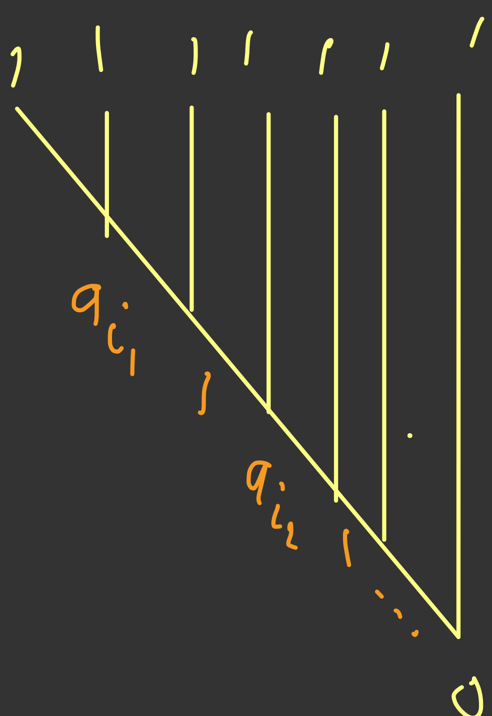
Notice: State spaces V_n are not $W^{\otimes n}$!

Basis for V_n:



$m_i \in \{0,1\}$

$\mathcal{H}_k = (\mathbb{C}^2)^k \hookrightarrow V_{2k+2}$



$\Rightarrow$ $i_j \in \{0,1\}$ freely chosen.

$|i_1 i_2 \cdots i_k\rangle$

$$\begin{tikzcd} \mathcal{H}_k \arrow[r, hook] \arrow[d, "U"'] & V_{2k+2} = \mathcal{H}_k \oplus \mathcal{H}_k^{\perp} \arrow[d, "\rho(\beta) = X"] \\ \mathcal{H}_k \arrow[r, hook] & V_{2k+2} \end{tikzcd}$$

X should appx U.

choose $\beta \in B_{2k+2}$ s.t. $|X - U \oplus I_{\mathcal{H}_k^{\perp}}| < \varepsilon$

There is a poly alg. to find β.

Approximating Jones Poly'l Evaluations.

(a version of) The Jones poly'l is derived from

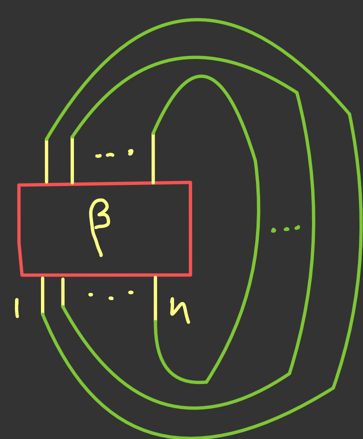
MTC $SU(2)_k$: from $U_q sl_2$ at $q = e^{\pi i/(k+2)}$.

Simple objects :

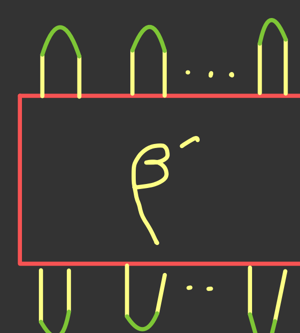
dims

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

Alexander's Th^m: K a knot/link, $\exists\ \beta \in B_n$ s.t.

$K \simeq$ 

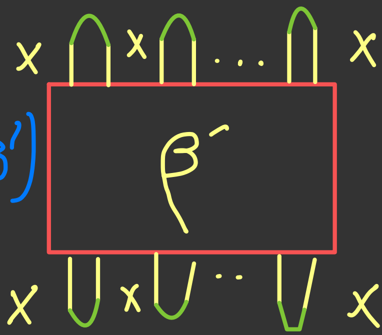
n is a poly'l in # crossings in K.

Joan Birman: Also true for plat closure: $K \simeq$ 

$\beta \in B_{2m}$

Jones polyl

regard K as a morphism in $SU(2)_k$

$J(K,q) = f(\beta')$  $\in \mathrm{End}(\mathbb{1}) \cong \mathbb{C}$.

adjust by braid index, etc

Thm: [Vertigan] Exact computation of $J(K,q)$ at $q = e^{\pi i/(k+2)}$ is #P-hard (in # of crossings) for $k \neq 0, 1, 2, 4$. ($k=3$ is essentially fibonacci).

Thm: $\left| \frac{J(K,q)}{d^n} \right|$ ($d = q + q^{-1}$) is in BQP

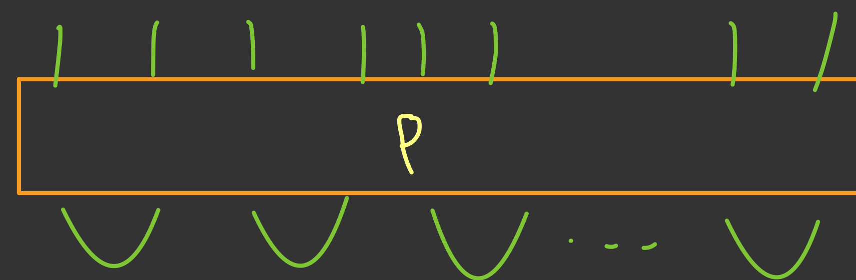
approximating.   braid index

Algorithm:

① Initialize $\cup \ \cup \ \cdots \ \cup = |\mathrm{cup}\rangle$

$\in \mathrm{Hom}(\mathbb{1}, X^{\otimes 2n})$

② Apply Braid $\rho(\beta)|\mathrm{cup}\rangle$



③ Measure against $\langle \mathrm{cap}| = \cap \ \cap \ \cdots \ \cap$

$\in \mathrm{Hom}(X^{\otimes 2n}, \mathbb{1})$

Probability of the outcome $\langle cap |$ is

$$\| \langle cap | \rho(\beta) | cup \rangle \|^2$$

Output: random variable $\tilde{Z}(\beta) \in \{0, 1\}$ for each run output 0 if outcome is $\langle cap |$.

$$\text{prob}(0) = \| \langle cap | \rho(\beta) | cup \rangle \|^2$$

$$= \left| \frac{J(K, q)}{d^n} \right|^2$$

qpt

Let $Z(\beta)$ = average of $\tilde{Z}(\beta)$ for some poly # of tries.

$$Z(\beta) \in [0, 1].$$

in $\frac{1}{\delta}$ $\delta > 0$ accuracy...

$$\mathrm{Prob}\left(\left| \left| \frac{J(K,q)}{d^n} \right|^2 - Z(\rho) \right| < \delta \right) \geq 3/4 .$$

$k = 2 \rightsquigarrow$ Ising Theory. $\{ \mathbb{1}, \sigma, \psi \}$

dim $= \sqrt{2}$

fermion

$\theta_\psi = -1$

Jones poly at $k=2 \longrightarrow$ Arf inv.

poly time classical alg.

Next time: Ising is not universal...

Is there an $\underline{F}$PRAS for $J(K,q)$?