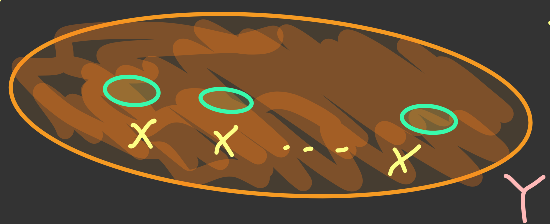
The following are the video link and slides for Eric Rowell’s Lecture 3.

https://livewarwickac-my.sharepoint.com/:v:/g/personal/u1972347_live_warwick_ac_uk/EfHo4_Y3InhEnLG6DjQwnZoBZmMgwOFqS9OdoWaNMvxdOg?e=BSdfZ4

Flavors of Anyons

Let $Y, X \in \mathcal{C}$ be a simple ($\mathrm{End}(X) \cong \mathbb{C}$) object.

$\mathcal{H}\Big($  $\Big) \overset{\rho}{\circlearrowleft} B_n$

X is

- Abelian if $\rho(B_n)$ is abelian for all n, Y.
- Non-abelian otherwise.

How to detect this?

Let $\mathcal{C}$ be UMTC (just fusion is fine), $\mathrm{Irr}(\mathcal{C}) = \{X_i\}$.

$X \in \mathcal{C}$: $\quad X \otimes X_i \cong \bigoplus_j m_{i,j}^X X_j$.

Define $[N_X]_{j,i} = m_{i,j}^X$ the fusion Matrix of X.

$K_0(\mathcal{C}) = (\mathrm{Irr}(\mathcal{C}), \otimes, \oplus, [\mathbb{1}])$ is a $\mathbb{Z}_+$-ring (commutative)

$[X] \longmapsto N_X$ is a rep. $K_0(\mathcal{C}) \to \mathrm{Mat}(\mathbb{N})$

$[X^*] \longmapsto N_X^T$

Defin: $\mathrm{FPdim}(X) = \max \mathrm{Spec}(N_X)$: the Frobenius–Perron dimension.

Some properties:

$\mathrm{FPdim}(X)$ is an algebraic integer

($\mathrm{char}_{N_X}(x)$ is monic)

$\mathrm{FPdim}(X) \in \mathbb{R}, \geq 1.$ (pf: N_X is not nilpotent)

$\mathrm{FPdim}(X) = \dim(X)$ for unitary $\mathcal{C}$. $\boxed{\text{Denote: } d_X}$

Define $\mathrm{FPdim}(\mathcal{C}) = \sum_{X_i} (d_{X_i})^2$

$\mathrm{FPdim}(\mathcal{C})$ is totally positive (all Galois conjugates are > 0)

$\dim(\mathrm{Hom}(\mathbb{1}, X^{\otimes n})) \approx c(d_X)^n$ assymptotically.

Ex: Fib: $V_n = \mathrm{Hom}(\mathbb{1}, \tau^{\otimes n})$ has $\dim(V_n) = f_n$

Fibonacci numbers. $d_\tau = \varphi = \frac{1+\sqrt{5}}{2}$

$f_n \approx c \cdot \varphi^n$

Suppose $d_X = 1$. Then $\dim(\mathrm{Hom}(Y, X^{\otimes n})) \in \{0, 1\}$.

(X is simple & so is $X^{\otimes n}$ $\forall n$).

So... B_n action is abelian (1-dim'l rep)

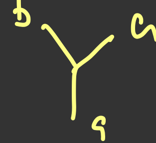
hence X is abelian.

Conversely? Yes!

Thm: [R-Wang] X is abelian iff $d_X = 1$.

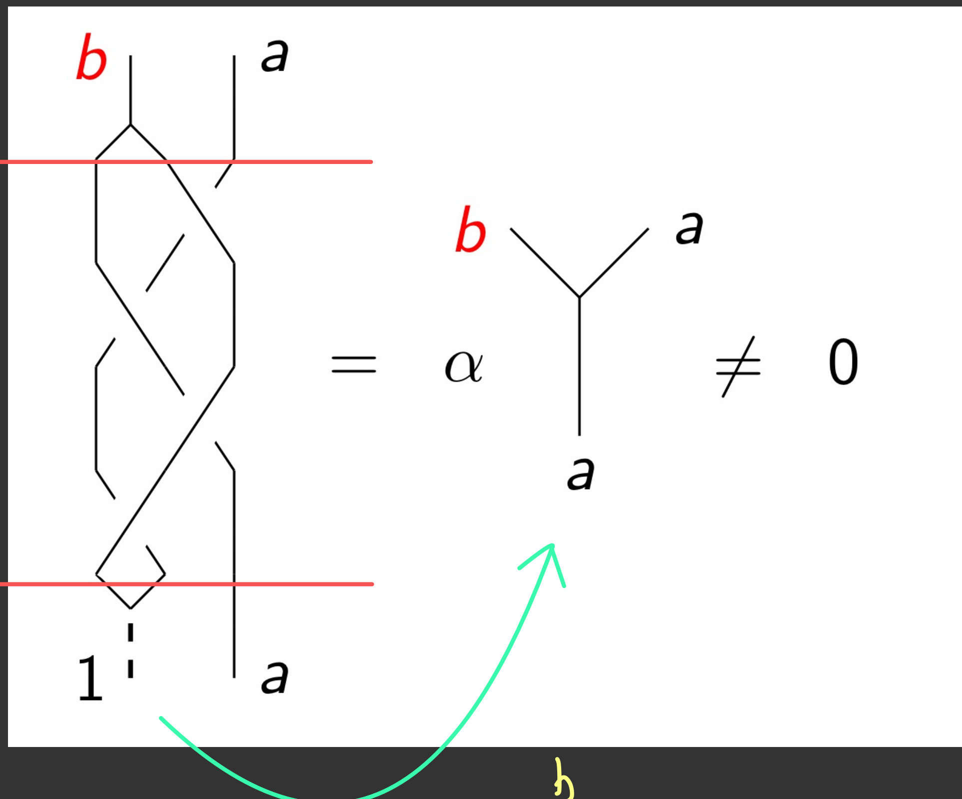
(Defn: $\mathcal{C}$ is pointed if all simples are abelian.)

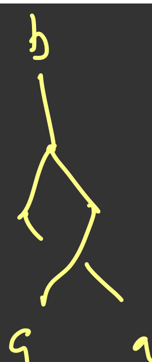
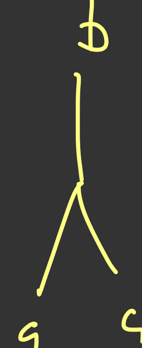
Assume $a \cong a^*$. If $d_a > 1$

$\exists\ b \neq \mathbb{1}$ s.t. $N_{aa}^{b} \neq 0$. So  $\in \mathrm{Hom}(a, a \otimes b)$

$(a \otimes a \not\cong \mathbb{1}.)$

$\rho(\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1)$



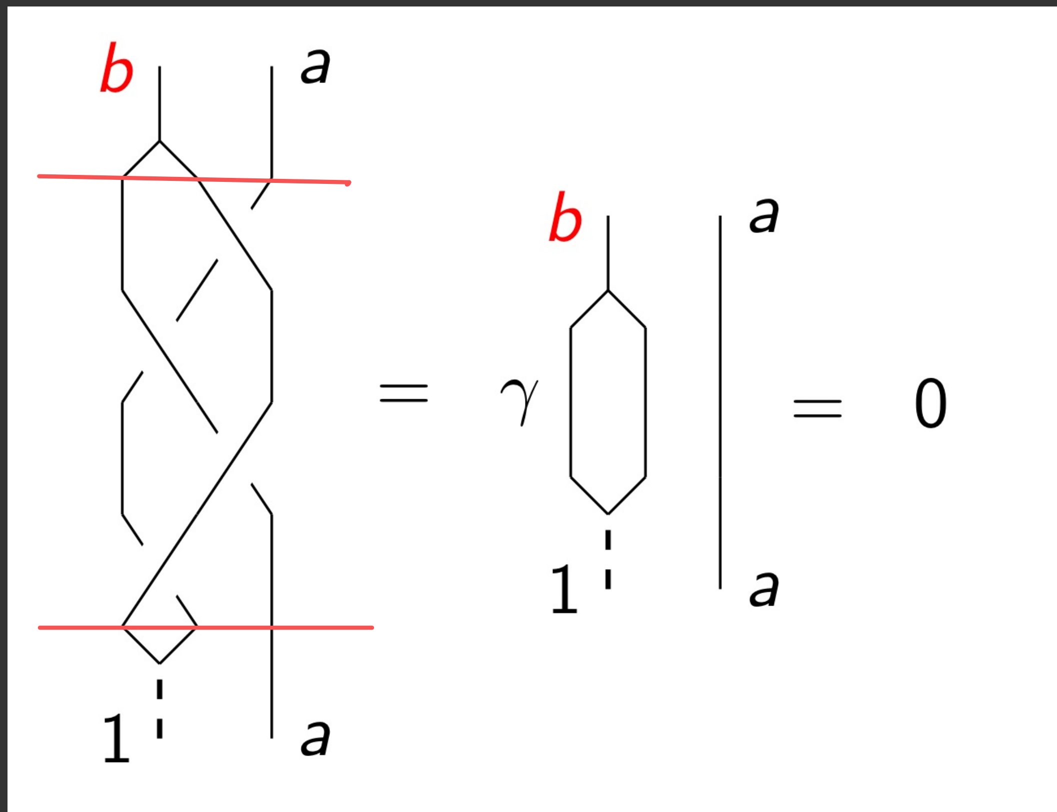
choice of basis.  $= R^{b}_{aa}$ 

s.t.

Suppose $\rho(\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1) = \gamma\cdot\mathrm{Id}$. $\gamma\in\mathbb{C}^\times$. (suppose a is abelian)

then:

$\gamma\cdot\mathrm{Id} \longrightarrow$



A contradiction!

Degeneracy of $a\ a \cdots a$ ground state $\Updownarrow$ a non-abelian

Since $d_a > 1 \iff \dim \mathrm{Hom}(X_b, X_a^{\otimes n}) \to \infty$

Given $X \in \mathcal{C}$ simple, set $H^X_{n,b} = \mathrm{Hom}(X^{\otimes n}, X_b)$

prev. lec. switched left vs. right action

So B_n acts as follows:

$$\mathbb{C}B_n \longrightarrow \mathrm{End}(X^{\otimes n})$$

$$\sigma_i \longmapsto I_X^{\otimes i-1} \otimes c_{X,X} \otimes I_X^{\otimes(n-i-1)}$$

$\mathrm{End}(X^{\otimes n})$ acts via $\circ$ on $\bigoplus_b H^X_{n,b}$

$$B_n \xrightarrow{\rho^X} \prod_b U(H^X_{n,b}).$$

Define: $a \leftrightarrow X_a = X$ has prop. F if $|\rho^X(B_n)| < \infty$ $\forall n$.

Generally, if a does not have prop. F then it yields universal TQC. Abuse notation: a is quasi-universal if $|\rho^X(B_n)| = \infty$. $(X = X_a)$.

Ex: G a finite gp, $\omega \in Z^3(G, U(1))$ a 3-cocycle. quasi-Hopf alg. $D^\omega G$ has $\mathrm{Rep}(D^\omega G)$ an MTC. Associated to Dijkgraaf-Witten TQFT. $\mathrm{FPdim}(\mathrm{Rep}\, D^\omega G) = |G|^2$. $d_X \in \mathbb{Z}$ $\forall X \in \mathrm{Rep}(D^\omega G)$.

Thm: [Etingof-Witherspoon-R] Each $X \in \mathrm{Rep}(D^\omega G)$ has prop. F.

Jones: $SU(2)_k$: $k \neq 1,2,4$ gen. obj. is quasi-universal

$k=1,2$ or 4 all objects have prop. F

Easy: abelian anyons have prop. F

Fibonacci $\dim(X) = \frac{1+\sqrt{5}}{2}$ is universal: braid group $\mathcal{B}_n$ image is dense in $SU(F_n) \times SU(F_{n-1})$	Ising $\dim(X) = \sqrt{2}$ is not universal: braid group $\mathcal{B}_n$ image is a finite group.
quasi-universal	prop. F

General pattern?

Conj. (2007, R-Naidu): $X \in \mathcal{C}$ simple has prop. F iff $(d_X)^2 \in \mathbb{Z}$.

Defn: $X \in \mathcal{C}$ simple is weakly integral if $(d_X)^2 \in \mathbb{Z}$.

$\boxed{\text{Conj: weakly integral iff prop. F.}}$

A major source of examples:

$\mathfrak{g}$ simple Lie alg. $q = e^{\pi i/l}$ a root of unity

$U_q \mathfrak{g} \rightsquigarrow \boxed{\mathcal{C}(\mathfrak{g}, l)}$ a modular category.

(usually, some restrictions)

Also use affine Kac-Moody $\to$ level preserving $\otimes$

Interesting case: $SO(N)_2$. ($U_q\,so(N)$, $q = e^{\pi i/2N}$ N odd...)

Simple objects have $d_X \in \{1, 2, \sqrt{N}, \sqrt{\tfrac{N}{2}}\}$ if N even.

(called Metaplectic anyons)

<u>Thm</u>: [R-Wenzl] All $X \in SO(N)_2$ have prop. F.

<u>Thm</u>: (Various) If $X \in \mathcal{C}(g, \ell)$ is not weakly integral then X is quasi-universal.

<u>Thm</u>: (Green-Nikshych). If $\mathcal{C}$ is weakly group-theoretical then all $X \in \mathcal{C}$ have prop. F.

Weakly Group Theoretical: Obtained from pointed or Ising categories via symmetry gauging (defined later).

<u>Conj:</u> (Drinfeld) Weakly Group Theoretical iff weakly integral.

Upshot: We can conjecturally characterize sources of Universal TQC by a simple measurement: $(d_X)^2$.

One More connection

Perspective: Fault-tolerance of TQC is due to non-locality. Simulating TQC on QCM uses a hidden locality ($\mathcal{S}_n \oplus \mathcal{S}_n^{\perp} = Y^{\otimes(n-1)}$)

non-comp. space

When is there direct locality?

Ex: $\mathrm{Rep}(D^{\omega}G)$: objects $\boxed{\text{are}}$ vector spaces!

2 actions of B_n: on $V^{\otimes n}$ & on $\bigoplus_{Y} \mathrm{Hom}(V^{\otimes n}, Y)$

these give essentially the same reps!

Local B_n reps? Yang–Baxter operators!

(R, W) $\quad R \in \mathrm{Aut}(W^{\otimes 2})$ s.t.

$\sigma_i \longmapsto I^{\otimes i-1} \otimes R \otimes I^{\otimes (n-i-1)}$ $\quad$ gives a B_n rep.

(check for $n=3$)

Braided Vector space.

A **localization** of a sequence of $\mathcal{B}_n$-reps. (ρ_n, V_n) is a braided vector space (R, W) and injective algebra maps $\tau_n : \mathbb{C}\rho_n(\mathcal{B}_n) \to \mathrm{End}(W^{\otimes n})$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C}\mathcal{B}_n \arrow[d, "\rho_n"] \arrow[dr, "\rho^R"] & \\
\mathbb{C}\rho_n(\mathcal{B}_n) \arrow[r, "\tau_n"] & \mathrm{End}(W^{\otimes n})
\end{tikzcd}$$

$X \in \mathcal{C}$ is localizable if $\exists$ a localization of

$(\mathcal{C}_n^X, H_n^X)$.

Ising σ is localizable

Fibonacci τ is not localizable...

Conjecture Anyons are either

localizable		non-localizable
weakly integral	OR	non-weakly integral
prop. F		quasi-universal
classical invariants		hard invariants

In fact, to prove $X \in SO(N)_2$ ($d_X = \sqrt{N}$ or $\sqrt{N/2}$) is prop F.

We show: X is localizable w/ (take N odd)

$$R = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} q^{j^2} U^j, \quad q = e^{2\pi i/N}, \quad U e_i \otimes e_j = q^{j-i} e_{i+1} \otimes e_{j+1}$$

which gives finite B_n images.

To verify universal from quasi-universal:

Use Lie Theory: $\rho : B_n \to U(V)$ irred.

what is $\overline{\rho(B_n)}$?

Look at eigenvalues of $\rho(\sigma_i)$. $\rightsquigarrow$ N-eigenvalue problem. [FLW, LRW]...

A *pair* (G, V) consists of a compact Lie group G and a faithful irreducible complex representation $\rho\colon G \to \mathrm{GL}(V)$. Let N be a positive integer. We say a pair (G, V) satisfies the $\boxed{N\text{-}eigenvalue\ property}$ if there exists a *generating element*, i.e., an element $g \in G$ such that the conjugacy class of g generates G topologically and the spectrum X of $\rho(g)$ has N elements and satisfies the *no-cycle property*: for all roots of unity ζ_n, $n \geq 2$, and all $u \in \mathbb{C}^\times$,

$$u\langle\zeta_n\rangle \not\subset X \tag{1.1}$$

Set $G = \overline{\rho(B_n)}$, $g = \rho(\sigma_1)$ ...

Classify pairs w/ N-eigenvalue prop.

<u>Ex:</u> τ fib. anyon: $V_n = \mathrm{Hom}(\mathbb{1}, \tau^{\otimes n})$

$\dim(V_n) = f_n$ $\quad \rho : B_n \to U(V_n)$

$\rho(\sigma_1)$ has 2-eigenvalues: $\{-1, \varphi\}$ $\varphi^5 = 1$.