The following are the video link and slides for Eric Rowell’s Lecture 4.

https://livewarwickac-
my.sharepoint.com/personal/u1972347_live_warwick_ac_uk/_layouts/15/onedrive.
aspx?id=%2Fpersonal%2Fu1972347%5Flive%5Fwarwick%5Fac%5Fuk%2FDocu
ments%2FAGQFT%2Frowell%2Dkias%2D4%2Emp4&parent=%2Fpersonal%2Fu
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UzAyWlBoaHNlN0dYSTg3c0J6SFhEQzlrdHJ2djh2X25XRmF1WnlBP3J0aW1lP
VN5ZXhfNDlZMlVn

Towards a Periodic Table of TPMs.

(bosonic) 2D TPMs $\longleftrightarrow$ UMTCs

$\boxed{\text{Spherical non-degenerate braided fusion category}}$

classify UMTCs? (up to ... ?)

- Constructions
  - "Natural"
  - New from old
- Witt gp perspective
- By rank
- Ultimately: up to braided monoidal equiv.

Natural Constructions

- Quantum gps: $\mathcal{C}(\mathfrak{g}, \ell)$

$$q = e^{\pi i/\ell}$$

$$\mathfrak{g} \rightsquigarrow U\mathfrak{g} \rightsquigarrow U_q\mathfrak{g} \rightsquigarrow \mathrm{Rep}(U_q\mathfrak{g}) \rightsquigarrow \mathcal{C}(\mathfrak{g}, \ell)$$

- Metric Gps: fin. abelian gp A, & non-deg. quad. form: $Q: A \to U(1)$.

$\mathcal{C}(A, Q)$. objects: $a \in A$, $\otimes$: mult in A,

$\mathrm{Hom}(a,b) = \delta_{a,b}\,\mathbb{C}(\mathrm{Id}_a)$ $a,b \in A$. Twists:

$\theta_a = Q(a)$. $d_a = 1$.

Th^m: Any pointed UMTC is of the form $\mathcal{C}(A, Q)$

Ex: $\mathcal{C}(\mathbb{Z}_p, Q_j)$ $\quad Q_j(a) = e^{2\pi i a^2 j/p}$ $\quad$ p odd.

Ex: $\mathcal{C}(\mathbb{Z}_2, 1 \mapsto i)$ Semion Theory

New From Old.

- If $\mathcal{C}, \mathcal{D}$ are MTCs the Deligne prod. $\mathcal{C} \boxtimes \mathcal{D}$ is also an MTC. "Direct prod."
  objects: (X, Y), Morphisms (f, g) & $\oplus$...

physically: bilayer construction.

Th^m: If $\mathcal{C} \subseteq \mathcal{D}$ MTCs then $\mathcal{D} \cong \mathcal{C} \boxtimes \mathcal{C}'$ for some MTC $\mathcal{C}'$

(factorization...)

0 Boson Condensation. G finite gp.

$\mathrm{Rep}(G)$ is a symmetric braided fusion cat:

$$c_{X,Y} \circ c_{Y,X} = \mathrm{Id}_{Y \otimes X} \quad \text{for all } X, Y \in \mathrm{Rep}(G)$$

Symmetric is "highly degenerate". In MTC

$$\{X \text{ simple} : c_{XY} c_{YX} = \mathrm{Id} \ \forall Y\} = \{\mathbb{1}\}.$$

(generally: Symmetric $\longleftrightarrow$ Super-Tannakian)

$\mathcal{S}$ is Tannakian if $\mathcal{S} \cong \mathrm{Rep}(G)$, some G.

$\hookrightarrow$ bosonic

If $\mathcal{S} \subseteq \mathcal{C}$ Tannakian subcat. of MTC, ($\mathcal{S} \cong \mathrm{Rep}(G)$)

may condense $\mathcal{S}$: $\mathcal{C}_G$ has same obj. as $\mathcal{C}$

but more morphisms.

Let $\mathrm{Fun}(G) \cong A \in \mathcal{C}$ an algebra object.

$$\boxed{\mathrm{Hom}_{\mathcal{C}_G}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(A \otimes X, Y)}$$ so: $A \longmapsto \mathbb{1}_{\mathcal{C}_G}$

$\mathcal{C}_G$ is G-graded: $\mathcal{C}_G \cong \bigoplus_{g \in G} \mathcal{D}_g : \mathcal{D}_g \otimes \mathcal{D}_h \subseteq \mathcal{D}_{gh}$

$\dim(\mathcal{D}_e) = \dfrac{\dim(\mathcal{C})}{|G|^2}$

$(\mathcal{C}_G)_e = \mathcal{D}_e$ is again an MTC.

Boson condensation of $\mathcal{C}$. A phase transition.

More generally: condense any connected étale algebra.

Idea: $A \in \mathcal{C}$ an alg. object: $m : A \times A \to A$ ...

$u : \mathbb{1} \longrightarrow A$

$\mathcal{C}_A$: A-modules in $\mathcal{C}$, $(\mathcal{C}_A)^0$ local A-modules

$\boxed{(\mathcal{C}_A)_0 \text{ is an MTC.}}$

braid trivially w/ A

Ex: $\mathrm{Rep}(D^\omega G) \supseteq \mathrm{Rep}(G) \underset{\text{condense}}{\rightsquigarrow} \mathrm{Vec}$.

- Symmetry Gauging. $\mathcal{C}$ MTC, $G \xrightarrow{\rho} \mathrm{Aut}^{br}_{\otimes}(\mathcal{C})$:

finite group G acting by braided $\otimes$-autoequivalences.

1. Extend/add G-defects: $\mathcal{D} = \bigoplus_g \mathcal{D}_g$: $\mathcal{D}_e = \mathcal{C}$.

2. G-equivariantize $\mathcal{D}^G$ ← an MTC: G-gauging

Cohomological obstructions/choices to Gauging...

Existence of fusion? Associative?

↳ choices ↳ choices

If so... $\mathcal{D}^G \supseteq \mathrm{Rep}(G)$ & $[(\mathcal{D}^G)_G]_e \simeq \mathcal{C}$.

So reverse process as Condensation.

Also a phase transition.

Ex: $\mathbb{Z}_2 \to \mathrm{Aut}^{br}_{\otimes}(\mathcal{C}(\mathbb{Z}_3, Q))$ $\quad Q(a) = e^{2\pi i a^2/3}$.

action is by $1 \leftrightarrow 2$, $0 \circlearrowleft$ on objects.

1. $\mathbb{Z}_2$-extension: $\mathcal{C}(\mathbb{Z}_3, Q) \oplus \mathcal{M}$: $\mathcal{M} = \{m\}$.

$\dim(m) = \sqrt{3}$ $\quad m \otimes m = 1 \oplus a \oplus a^{-1}$. (TY-cat)

2. $\mathbb{Z}_2$-equiv. : $\cong SU(2)_4$ (for one choice).

Has dims: $1, 1, 2, \sqrt{3}, \sqrt{3}$ (5 simples).

- Zesting: Combines $\boxtimes$ & boson condensation...

Question: Can we Gauge more general symmetries?

I.e. reverse "local A-module" construction.

∘ Drinfeld Center: From any spherical fusion cat $\mathcal{B}$ get $Z(\mathcal{B})$ an MTC. $(\dim(Z(\mathcal{B})) = \dim(\mathcal{B})^2)$

objects are pairs $(X, \gamma_{X,-})$ $X \in \mathcal{B}$, $\gamma_{X,Y} : X \otimes Y \simeq Y \otimes X$ for all $Y \in \mathcal{B}$. "half-braiding"

Ex: $Z(\mathrm{Rep}\, G) \cong \mathrm{Rep}(DG)$.

Ex: If $\mathcal{C}$ is modular, $Z(\mathcal{C}) \cong \mathcal{C} \boxtimes \mathcal{C}^{rev}$

reverse braiding

$\tilde{c}_{X,Y} = c^{-1}_{Y,X}$.

Witt Gp. (Davydov, Müger, Nikshych, Ostrik)

Modular categories form an abelian gp:

1. $\mathcal{C} \sim \mathcal{D}$ if $\exists$ fusion cat $\mathcal{A}$ s.t.

$\mathcal{C} \boxtimes \mathcal{D}^{rev} \cong Z(\mathcal{A})$. Equiv. rel on MTCs.

2. $[\mathcal{C}]\cdot[\mathcal{D}] = [\mathcal{C}\boxtimes\mathcal{D}]$.

3. Id: $[Z(\mathcal{A})]$. $[\mathcal{C}]^{-1} = [\mathcal{C}^{rev}]$.

Called the Witt gp: Abelian, ∞ rank, torsion is infinite 2-gp. of exponent 32.

Boson Condensation, local A-module cats, Gauging ALL preserve Witt class.

<u>Conj:</u> (DMNO) $\mathcal{C}(\mathfrak{g}, \ell)$ generate the Witt gp.

"All MTCs come from Quantum Gps."

Classification By rank.

- Ocneanu Rigidity: for fixed $\{N_{ij}^{k}\}$ (at most) finitely many braided fusion cats.

- Any braided fusion category has finitely many spherical structures.
- Verlinde: S_{ab} determines N_{ij}^{k}.

Upshot: Classify Modular Data:

Definition 2.7. Let $S, T \in \mathrm{GL}_r(\mathbb{C})$ and define constants $d_j := S_{0j}$, $\theta_j := T_{jj}$, $D^2 := \sum_j d_j^2$ and $p_\pm = \sum_{k=0}^{r-1} (S_{0,k})^2 \theta_k^{\pm 1}$. The pair (S, T) is an **admissible modular data** of rank r if they satisfy the following conditions:

dim($\mathcal{C}$)

diagonal

(i) $d_j \in \mathbb{R}$ and $S = S^t$ with $S\overline{S}^t = D^2\,\mathrm{Id}$. $T_{i,j} = \delta_{i,j}\theta_i$ with $N := \mathrm{ord}(T) < \infty$.

(ii) $(ST)^3 = p^+ S^2$, $p_+ p_- = D^2$ and $\frac{p_+}{p_-}$ is a root of unity.

$\mathbb{F}_S := \mathbb{Q}(S_{ij})$
$\mathbb{F}_T := \mathbb{Q}(T_{ij})$.

(iii) $N_{i,j}^k := \frac{1}{D^2} \sum_{a=0}^{r-1} \frac{S_{ia} S_{ja} \overline{S_{ka}}}{S_{0a}} \in \mathbb{N}$ for all $0 \le i, j, k \le (r-1)$.

(iv) $\theta_i \theta_j S_{ij} = \sum_{k=0}^{r-1} N_{i^*j}^k d_k \theta_k$ where i^* is the unique label such that $N_{i,i^*}^0 = 1$.

(v) Define $\nu_n(k) := \frac{1}{D^2} \sum_{i,j=0}^{r-1} N_{i,j}^k\, d_i d_j \left(\frac{\theta_i}{\theta_j}\right)^n$. Then $\nu_2(k) = 0$ if $k \neq k^*$ and $\nu_2(k) = \pm 1$ if $k = k^*$. Moreover, $\nu_n(k) \in \mathbb{Z}[e^{2\pi i/N}]$ for all n, k.

(vi) $\mathbb{F}_S \subset \mathbb{F}_T = \mathbb{Q}_N$, $\mathrm{Gal}(\mathbb{F}_S/\mathbb{Q})$ is isomorphic to an abelian subgroup of $\mathfrak{S}_r$ and $\mathrm{Gal}(\mathbb{F}_T/\mathbb{F}_S) \cong (\mathbb{Z}/2\mathbb{Z})^\ell$ for some integer ℓ.

(vii) (Cauchy Theorem, [26, Theorem 3.9]) The prime divisors of D^2 and N coincide in $\mathbb{Z}[e^{2\pi i/N}]$. $= \mathbb{Z}[\zeta_N]$

Tools we can use:

- Galois theory: S_{ij}, T_{ij} lie in Abelian extn of $\mathbb{Q}$.
- Matrix analysis: N_x diagonalized by S.
- $SL(2,\mathbb{Z})$ rep. theory: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mapsto S$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T$

  proj. rep of $SL(2,\mathbb{Z})$.

  Factors over $SL(2, \mathbb{Z}/N)$ a finite gp
  where $N = \mathrm{ord}(T)$.

- Algebraic # Thy: $\mathbb{Z}[\zeta_N]$ a Dedekind Domain,

  S_{ij} alg. integers.
- Algebraic geom: Poly'l equations (use Gröbner...)

Putting these together... Let $G = \mathrm{Gal}(\mathbb{F}_S/\mathbb{Q})$

$$S = \begin{bmatrix} 1 & d_1 & \cdots & d_{r-1} \\ d_1 & & & \\ \vdots & & S_{ij} & \\ d_{r-1} & & & \end{bmatrix} \qquad S N_i S^{-1} = \begin{bmatrix} d_i & & & 0 \\ & \ddots & \frac{S_{ij}}{d_j} & \\ 0 & & & \ddots \end{bmatrix}$$

Let $\sigma \in G$. $\left\{ \frac{S_{ij}}{d_j} : j = 0, \dots, r-1 \right\}$

are eigenvalues of N_i: $\sigma\left(\frac{S_{ij}}{d_j}\right) = \frac{S_{ij'}}{d_{j'}}$

Independent of i

So $G \hookrightarrow \mathfrak{S}_r$ (permutes columns)

$\sigma \longmapsto \hat{\sigma}$

From this: $\sigma(S_{ij}) = \pm S_{\hat{\sigma}(i)\hat{\sigma}^{-1}(j)}$

Thm: S_{ij} lie in a cyclotomic field

So $G = \mathrm{Gal}(\mathbb{F}_S/\mathbb{Q})$ an abelian subgp of $\mathcal{C}_r$.

Ex: $r = 2$ $\begin{bmatrix} 1 & d_1 \\ d_1 & S_{11} \end{bmatrix}$ $\perp$ cols so $S_{11} = -1$

If $G = \langle e \rangle$ $d_1 \in \mathbb{Z}$.

If $G = \langle (0\,1) \rangle$ ... $d_1^2 + n d_1 \pm 1 = 0$

$\vdots$

Since $\rho : SL(2,\mathbb{Z}) \to GL_r(\mathbb{C})$ factors over

$$SL(2,\mathbb{Z}/N) \cong SL(2,\mathbb{Z}/p_1^{\alpha_1}) \times \cdots \times SL(2,\mathbb{Z}/p_k^{\alpha_k})$$

by CRT, can decompose...

Low dim reps. are classified so in low ranks

we write $\hat{S} = \begin{bmatrix} \ddots & & \\ & \boxed{} & \\ & & \ddots \end{bmatrix}$ $\hat{T} = \begin{bmatrix} \ddots & & 0 \\ & \delta_{ij} t_{ii} & \\ 0 & & \ddots \end{bmatrix}$

& look for F (orthogonal) s.t.

$$S = F \hat{S} F^{-1} \quad \& \quad F \hat{T} F^{-1} = T$$

satisfy all conditions.

We can classify up to rank 6 up to fusion rules
(& often up to br. $\otimes$ equiv.)

Below: $SU(N)_k \cong PSU(N)_k \boxtimes \mathcal{C}(\mathbb{Z}/_N, Q)$ for $(N,k)=1$.

$PSO(8)_3 \subseteq SO(8)_3$ is mod. sub cat.

Rank	representative (of fusion rules)
2	$PSU(2)_3$, $\mathcal{C}(\mathbb{Z}_2, Q)$
3	$PSU(2)_5$, $\mathcal{C}(\mathbb{Z}_3, Q)$, $SU(2)_2$
4	$PSU(2)_7$, $\mathcal{C}(\mathbb{Z}_4, Q)$, $\mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, Q)$, $\boxtimes$ of 2
5	$PSU(2)_9$, $\mathcal{C}(\mathbb{Z}_5, Q)$, $SU(2)_4$, $PSU(3)_4$
6	$PSU(2)_{11}$, $SO(5)_2$, $PSO(5)_{\frac{3}{2}}$, $PSO(8)_3$, $(G_2)_3$ $\boxtimes$ of 2,3

Question: How many MTCs for fixed rank r?

Wang Conj. ('03) Finitely many.

('16) [Bruillard, Ng, R, Wang] Conjecture holds.

Uses: S-units. (Analytic # Thy).

pf outline:

1. $\mathrm{Ord}(T) = N$ bdd in terms of r.
2. A bd. on $\dim(\mathcal{C}) \Rightarrow$ bd. on N_{ij}^{k}.
3. Cauchy Th^{m} for MTCs: in $\mathbb{Z}[\zeta_N] = \mathbb{Z}[T_{ii}]$

$$\mathrm{Spec}\langle N \rangle = \mathrm{Spec}\langle \dim(\mathcal{C}) \rangle.$$

(this uses FS-indicators...)

4. $\dim(\mathcal{C}) = \sum_i d_i^2 \Rightarrow 1 = \sum_{i=0}^{r-1} \left(\frac{d_i^2}{\dim(\mathcal{C})} \right)$ (*)

5. $\frac{d_i^2}{\dim(\mathcal{C})}$ are S-units for $S = \mathrm{Spec}\langle N \rangle$.

$\leftarrow$ $d_i^2 \mid \dim(\mathcal{C})$

fractional ideal $\left\langle \frac{d_i^2}{\dim(\mathcal{C})} \right\rangle$ factor as prime ideals from S.

6. Evertse '84: finitely many sol'ns to S-unit eqn. (*).

7. Finite choices for $\dim(\mathcal{C})$, so fd.

Q.E.D.

Remarks: o S-unit eqns are like Diophantine eqns with a finite choice of primes.

o Does not give effective bd.: (quadruple exponential)

o Can be extended to rank-finiteness for Braided Fusion Categories (Morrison, Nikshych, Jones, R).