The following are the video link and slides for Eric Rowell's Lecture 5.

https://livewarwickac-my.sharepoint.com/:v:/g/personal/u1972347_live_warwick_ac_uk/Ea_inSwBckBIi-Wb0_alm4AB9zJR_-t1qmaj2aOVG5h26Q?e=FVfW3G

Further Topics.

1. Quantum Error Correcting codes: Embedding

$$\mathcal{H}_k = (\mathbb{C}^d)^{\otimes k} \xrightarrow{i} (\mathbb{C}^d)^{\otimes n} \xrightarrow[A_m]{} (\mathbb{C}^d)^{\otimes n} \xrightarrow{\pi} (\mathbb{C}^d)^{k}$$

$= \lambda \cdot \mathrm{Id}_{\mathcal{H}_k}$ for all m-local A_m, i.e.

$A_m = \mathrm{Id}^{a} \otimes U \otimes \mathrm{Id}^{b}$: U acts on m qudits.

"Corrects m-local errors".

Expect: ground state spaces of TPMs are QECCs. See Cui, et al & Qin-Wang...

2. Measurement Enhanced TQC

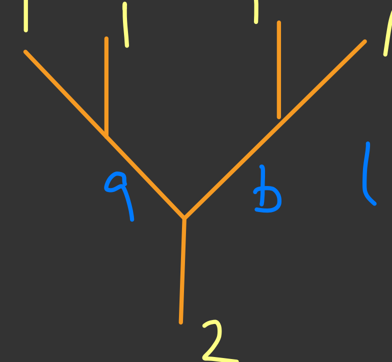
Metaplectic anyons: Labels $\{0,1,2,3,4\}$

$\dim(X_1) = \dim(X_3) = \sqrt{3}$ $\quad \dim(X_2) = 2.$ $\quad SU(2)_4$

$\dim(\mathbb{1}) = \dim(X_4) = 1$

Fusion: $1 \otimes 1 = 0 \oplus 2$ $\quad 2 \otimes 2 = 0 \oplus 2 \oplus 4$

Encode a qutrit:



$(a,b) \in \{(2,2), (0,2), (2,0)\}.$

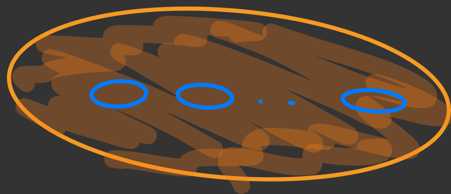
$|0\rangle, |1\rangle, |2\rangle$

Besides braiding add: Measure left pair to determine a. projects onto $\{|0\rangle, |2\rangle\}$. ← non-topological

This yields universal model. (Cui, Wang)

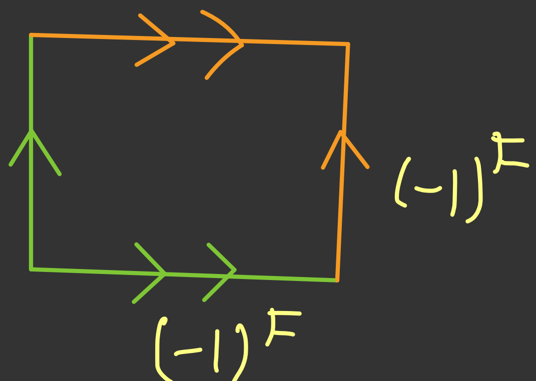
(adds a single qutrit gate ...)

3. Beyond bosonic 2D TPMs.



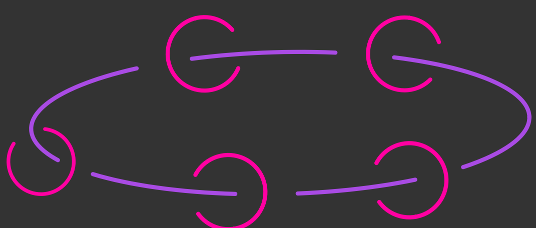
$\longrightarrow$ (2+1) TQFTs $\longleftrightarrow$ UMTCs

(Turaev...)



$\longrightarrow$ Spin TQFTs $\longleftrightarrow$ Super-MTCs

(Gaiotto-Kapustin)



$\longrightarrow$ (3+1) TQFTs $\longleftrightarrow$ Spherical fusion 2-categories

How to include fermions? Extra layer of non-locality.

3 choices: Super-Modular.
Spin-Modular
Fermionic Modular

Let $\mathcal{C}$ be a BFC (braided fusion cat).

$\mathcal{C}' := \langle \{ X : c_{XY} c_{YX} = \mathrm{Id} \ \forall Y \in \mathcal{C} \} \rangle$ subcat. gen'd by...

The symmetric or Müger center.

$\mathcal{C}'$ is symmetric.

Modular: $\mathcal{C}' \cong \mathrm{Vec} = \langle \mathbb{1} \rangle$ trivial.

Super-Modular/slightly degenerate: $\mathcal{C}' \cong$ sVec.

sVec: symmetric UBFC 2 simples: $\mathbb{1}$, f

$f^2 = \mathbb{1}$, $c_{f,f} = -\mathrm{Id}_{f\otimes f}$, $\theta_f = -1$, $\dim(f) = 1$.

$\mathbb{Z}_2$-graded vector spaces $S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

So super-modular contains one transparent fermion

Thm: Let $\mathcal{C}$ be a UBFC & $\mathcal{T} \subseteq \mathcal{C}'$ max'l Tannakian: $\mathcal{T} \cong \mathrm{Rep}(G)$. $\mathcal{C}_G$ is either modular or super-modular.

So: To classify UBFCs, study mod/super-mod.

MMEs: Suppose $\mathcal{B}$ super-mod. & $\mathcal{C}$ MTC w/ $\mathcal{B} \subseteq \mathcal{C}$. Then $\dim(\mathcal{C}) \geq 2\dim(\mathcal{B})$. When sharp?

$\mathcal{C}$ MTC, $\mathcal{B}$ a super-MTC. $\mathcal{C}$ is a minimal modular extension of $\mathcal{B}$ if $\mathcal{B} \subseteq \mathcal{C}$, $\dim(\mathcal{C}) = 2 \cdot \dim(\mathcal{B})$.

MMEs exist? Yes! Johnson-Freyd / Reutter ('21).

How many? Exactly 16. Lan-Kong-Wen

A MTC $\mathcal{C} \supseteq sVec$ is called Spin-modular.

Rank bounds? $\mathcal{B} \subseteq \mathcal{C}$ MME: $\frac{3\, rk(\mathcal{B})}{2} \leq rk(\mathcal{C}) \leq 2\, rk(\mathcal{B})$.

Kitaev: 16 fold way: $\mathcal{C} \supseteq sVec$ MMEs are

- 2 $\mathcal{C}(\mathbb{Z}_4, Q)$ & $\mathcal{C}(\mathbb{Z}_4, Q)^{rev}$
- 1 $TC = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, Q_1)$ twists: $1, 1, 1, -1$
- 1 $3F = \mathcal{C}(\mathbb{Z}_2 \times \mathbb{Z}_2, Q_2)$ twists: $1, -1, -1, -1$
- 8 Galois conjugates of Ising.
- 4 prods.

If $\mathcal{B}$ is super-MTC,

$$S_{\mathcal{B}} = \hat{S} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad T_{\mathcal{B}} = \hat{T} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(In partic: $rk(\mathcal{B})$ even) $\hat{S}$, $\hat{T}$ invertible but $\hat{T}$ ambiguous sign, however $\hat{T}^2$ is well-defined.

Do $\hat{S}$, $\hat{T}^2$ give a rep of something? Yes!

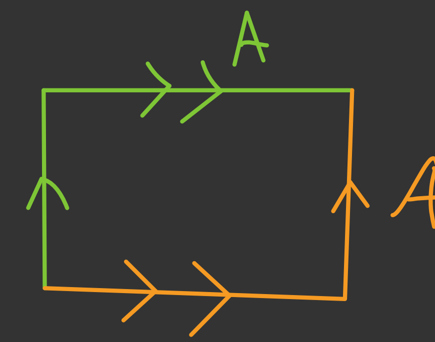
In $SL(2,\mathbb{Z}) = \langle s, t \rangle$ index 3 subgp:

$\langle s, t^2 \rangle = \Gamma_\theta$.

Topological Interpretation? Yes!

Consider spin structure on T^2

Anti-periodic on both cycles:



Subgp of $MCG(T^2)$ preserving spin-structure: Γ_θ.

Image?

Th^m: [Ng-Schauenburg] $SL(2,\mathbb{Z})$ rep. from MTC has congruence kernel $\supseteq \Gamma(N)$.

(all $\equiv I \bmod N$)

Th^m: (Bonderson, R, Zhang, Wang) (uses MME).

Also true for Γ_θ rep from Super-MTC

Classification?

Morrison-Jones-Nikshych-R: rank-finiteness for super-MTCs

Examples: ○ Split super-MTCs: $\mathcal{D} \boxtimes sVec$, $\mathcal{D}$ MTC.

○ Let f be any fermion in a spin-MTC.

$\langle f \rangle' := \langle \{ X : c_{X,f} c_{f,X} = Id \} \rangle$ is super-MTC. MME Th^m $\Rightarrow$ all of them.

Ex: $SU(4k+2)_{4m+2}$, $SO(2k+1)_{2m+1}$

$Sp(2r)_m$: $rm \equiv 2 \bmod 4$

$SO(2r)_m$: $r \equiv m \equiv 2 \bmod 4$

$(E_7)_{4m+2}$

} spin-MTCs

Complete classification for rk ≤ 6 (non-split):

$PSU(2)_2 = sVec$, $PSU(2)_6$, $PSU(2)_{10}$

rk: 2 4 6

Rank 8: at least 3: $PSU(2)_{14}$, $\langle f' \rangle \leq SO(12)_2$

& $[PSU(2)_6 \boxtimes PSU(2)_6]_{\mathbb{Z}_2}$. (propably complete)

Condense fermions?

Suppose we try to condense fermions. $F = \mathbb{1} + f$ is a non-commutative algebra.

$f \otimes X \cong X$ or $f \otimes X \not\cong X$ $\rightsquigarrow$ $\mathbb{Z}_2$-graded Hom spaces.

Not linear, super-linear.

$\downarrow$

pair of dims $(i|j)$

fermionic-MTC.

Ex: $PSU(2)_6$: rank 4 super-mod cat.

$\{\mathbb{1}, X, f, f\otimes X\} \longrightarrow \{\mathbb{1}, \bar{X}\}$ (objects)

$$\bar{X}^{\otimes 2} = \mathbb{1} + (1|1)\bar{X}$$

Super-MTC ⟶ Spin-MTC

↓

fermionic-MTC

↑

Moore-Read ...

which to use?

Loop-like excitations

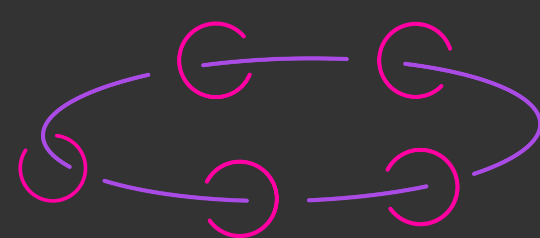
pts in $\mathbb{R}^3$ ... boring.

Loops in $\mathbb{R}^3$: interesting.

Any (3+1) TQFT: $M^3 \longmapsto \mathcal{H}(M^3)$

In analogy w/ 2D: motions of points in $\Sigma^2 \rightsquigarrow$ reps.

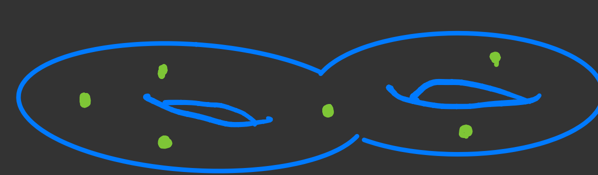
Motions of 1-manifolds in $M^3 \rightsquigarrow$ reps.



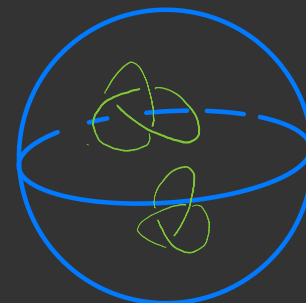
Motion Groups

Dahm '62, Goldsmith '80s

$N \subseteq M$ submanifold



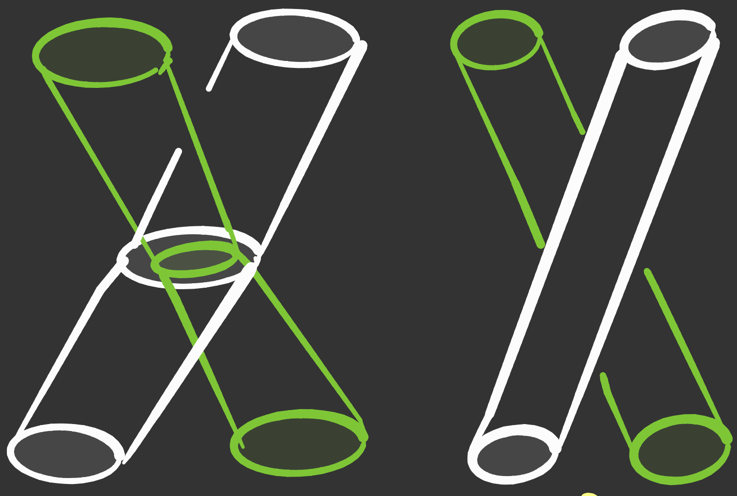
- A motion of N in M is an ambient isotopy $f_t(x)$ of N in M s.t.



1. $f_0 = Id_M$
2. $f_1(N) = N$ as a submanifold

- f is stationary if $f_t(N) = N \ \forall t$
- $f \simeq g$ if $g^{-1} \circ f \sim$ a stationary motion. ($\sim$: homotopic)
- $\mathcal{M}(M, N)$ motions $/ \simeq$.

Example 1 (McCool, Fenn-O'Roarke-Rimanyi, ...)



Loop Braid group

The **braid relations***:*

(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$

(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j|>1$,

the **symmetric group relations***:*

(S1) $s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$

(S2) $s_is_j = s_js_i$ *for* $|i-j|>1$,

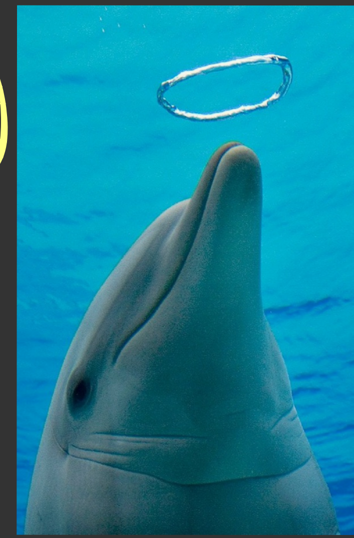
(S3) $s_i^2 = 1$

and the **mixed relations***:*

(L0) $\sigma_is_j = s_j\sigma_i$ *for* $|i-j|>1$

(L1) $s_is_{i+1}\sigma_i = \sigma_{i+1}s_is_{i+1}$

(L2) $\sigma_i\sigma_{i+1}s_i = s_{i+1}\sigma_i\sigma_{i+1}$



$$LB_n = \mathcal{M}(D^3, \sqcup^n S^1) \supseteq B_n$$

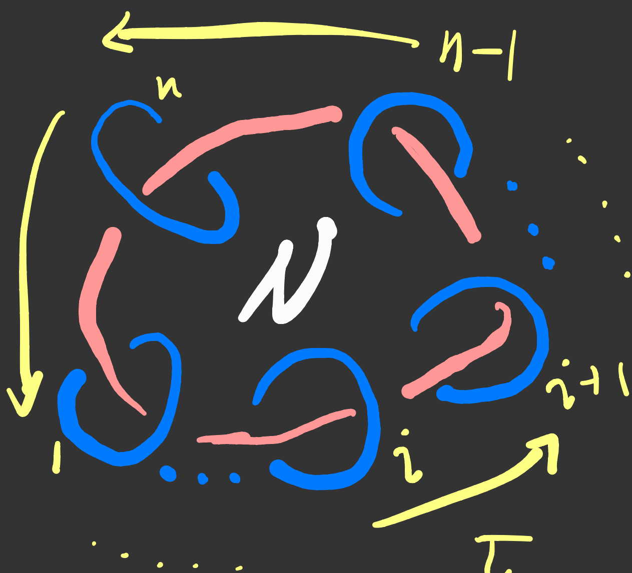
heuristic defn

Example 2 (Bellingeri–Bodin)

$\mathcal{M}(S^3, N) = NB_n \supseteq B_n$

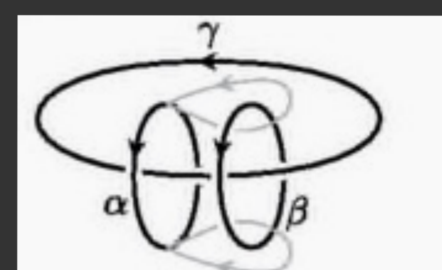
(B1) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$
(B2) $\sigma_i\sigma_j = \sigma_j\sigma_i$ *for* $|i-j| \neq 1 \pmod{n}$,
(N1) $\tau\sigma_i\tau^{-1} = \sigma_{i+1}$ *for* $1 \leq i \leq n$
(N2) $\tau^{2n} = 1$

Here indices are taken modulo n, with $\sigma_{n+1} := \sigma_1$ *and* $\sigma_0 := \sigma_n$.



Necklace Braid Group

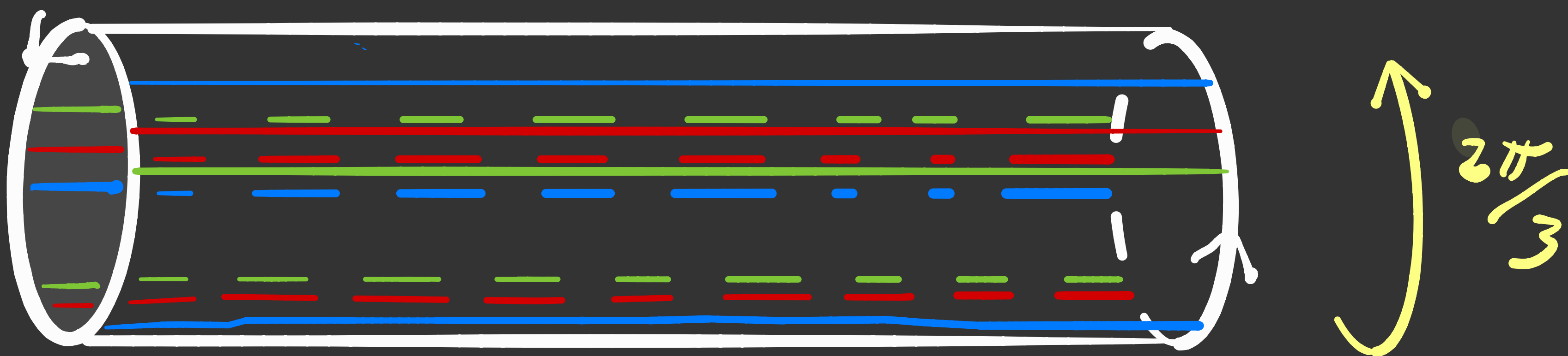
Levin-Wang
PRL '14
TPMs →



Example 3

(Goldsmith, Qiu–Wang)

Torus Links $TL(np, nq)$



σ_i : interchange $i, i+1$ components

r_i : rotate ith comp. by $2\pi/p$

→ satisfy B_n rels.!

Work w/ P. Martin & others...

Study reps of Motion gps.

Key: usually B_n is a subgp. So lift known reps.

1. Getting reps directly from (3+1) TQFT is hard...

2. We can get non-s.s. reps, which could be more powerful...

Thank You!