The following are the video link and slides for Hiraku Nakajima's Lecture 1.

https://livewarwickac-my.sharepoint.com/:v:/g/personal/u1972347_live_warwick_ac_uk/EfwMgL-QlXFNn52TlRuXTbUBZA3dqXvCx0Vlr6rQaj03Hg?e=KpKwrt

w Braverman, Finkelberg

$\left(\begin{array}{l} G: \text{cpx reductive grp} \qquad G_C : \text{max. cpt subgrp} \\ N: \text{finite dim. rep. of } G \\ M = N \oplus N^* \quad : \text{symplectic rep. of } G \\ \qquad\qquad (\text{cotangent type}) \end{array}\right.$

$\rightsquigarrow$ 3d $N=4$ SUSY gauge theory
based on $\left(\begin{array}{l} \bullet \text{ a connection } A \text{ on } P \xrightarrow{G_C} M^3_{,g} \\ \bullet \text{ spinor/forms} \\ \qquad \text{with values in } P \times^{G_C} M \end{array}\right.$

$\rightsquigarrow$ Higgs/Coulomb branches

$\mathcal{M}_H \equiv \mathcal{M}_H(G, N) \qquad \mathcal{M}_C \equiv \mathcal{M}_C(G, N)$

hyperKähler manifolds
possibly with singularities

Roughly   gauge theory

$\overset{\sim}{\text{"equiv."}}$ σ-model $M^3 \rightarrow \circledast$
                                whose
                            target space
                            $\mathcal{M}_C \cup \mathcal{M}_H$

$\mathcal{M}_H$ $\underline{\underline{\text{does not}}}$ receive "quantum correction"

$\mathcal{M}_C$ $\underline{\underline{\text{does}}}$ receive   "

BFN : mathematically rigorous definition
of $\mathcal{M}_C$ as an algebraic variety

✳ FI parameter
$$\zeta \in \text{Hom}(\mathbb{C}^\times, \pi_1(G)^\wedge)$$
$\hookleftarrow$ Pontryagin dual
$$\cong \text{Hom}_{\text{grp}}(G, \mathbb{C}^\times)$$
$$\pi_1(\mathbb{C}^\times)^\wedge = (\mathbb{C}^\times)^\vee$$
dual

ex. • $G = GL_n(\mathbb{C})$ or its products
$$\longrightarrow \pi_1(G) = \mathbb{Z}, \mathbb{Z}^N$$
• $G = SL_n(\mathbb{C}) \longrightarrow \pi_1(G) = 1$

✳ flavor symmetry
$$\widetilde{G} \triangleright G \qquad \widetilde{G} \curvearrowright N$$
normal subgrp $\qquad \cup G$
$$G_F = \widetilde{G}/G \qquad \text{flavor symmetry group}$$
$\cup$
T_F max torus

mass parameter $\quad m \in \text{Hom}(\mathbb{C}^\times, T_F)$

§ Higgs branch

$$\mathcal{M}_H \equiv \mathbb{M} /\!/\!/ G \qquad \text{symplectic reduction}$$

$$\mu : \mathbb{M} \longrightarrow \mathfrak{g}^* \qquad \text{moment map}$$

$$\langle \underset{\underset{\mathbb{M}}{\uparrow}}{\mu(x)}, \underset{\underset{\mathfrak{g}}{\uparrow}}{\xi} \rangle = \frac{1}{2} \underset{\underset{\text{symp. form on } \mathbb{M}.}{\big\downarrow}}{\omega}(\xi x, x)$$

$$\mathbb{M} /\!/\!/ G = \mu^{-1}(0) /\!/ G \qquad \text{categorical}$$
$$= \operatorname{Spec}\left(\mathbb{C}[\mu^{-1}(0)]^G \right) \qquad \text{quotient}$$

$$\underset{\mathbb{C}^\times}{\overset{T_F}{\underset{m}{\nearrow}}} \overset{\subset}{\underset{\|}{G_F}} \overset{\curvearrowright}{\widetilde{G}/G} \mathcal{M}_H$$

✲ FI parameter $\quad \zeta \in \operatorname{Hom}_{grp}(G, \mathbb{C}^\times)$

$$\mathbb{C}[\mu^{-1}(0)]^{G, \zeta^n} = \{ f \in \mathbb{C}[\mu^{-1}(0)] \mid g^* f = \zeta^n(g) f \}$$
$$\text{relative invariants}$$

$$\underset{\mathbb{C}[\mu^{-1}(0)]^G}{\overset{\subset}{}} \quad \bigoplus_{n \geq 0} \mathbb{C}[\mu^{-1}(0)]^{G, \zeta^n} \qquad \text{graded algebra}$$

$$\mathcal{M}_H^\zeta := \mu^{-1}(0) /\!\!/_{\zeta} G = \operatorname{Proj}\left(\quad \nearrow \quad \right) = \mu^{-1}(0)^{\zeta-ss} /\!\!\sim (S\text{-equiv.})$$

$$\mathcal{M}_H^\zeta \longrightarrow \mathcal{M}_H \qquad \text{proj. morphism} \qquad \underset{}{\overset{\|}{\mu^{-1}(0)^{\zeta-st}/G}} \overset{\| \text{ nice}}{} \text{can}$$

(Kähler parameter)

$$d\zeta \in \text{Hom}_{\text{Lie alg}} \overset{//\text{Lie } G}{(\mathfrak{g}, \mathbb{C})} \subset \mathfrak{g}^*$$

$$\zeta_{\mathbb{C}} \qquad \text{fixed by coadjoint action}$$

$$\mu^{-1}(\zeta_{\mathbb{C}}) // G = \mathcal{M}_H^{\zeta_{\mathbb{C}}}$$

$$\mu^{-1}(\zeta_{\mathbb{C}}) \underset{\zeta}{//} G = \mathcal{M}_H^{\zeta, \zeta_{\mathbb{C}}}$$

<u>Remarks</u> (1) hyperkähler quotient
(instead of GIT quotient)

(2) If $G \curvearrowright \mu^{-1}(0)^{\zeta\text{-stable}}_{(\text{sem})}$ point
is free,

$$\mu^{-1}(\zeta_{\mathbb{C}})^{\zeta\text{-st.}} \quad \text{and} \quad \mu^{-1}(\zeta_{\mathbb{C}}) \underset{\zeta}{//} G$$

are of expected dim,

$$\dim \mathbb{M} - 2 \dim G.$$

Example
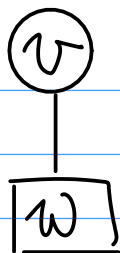
$v, w \in \mathbb{Z}_{\geq 0}$

(quiver gauge theory)

$$V = \mathbb{C}^v \quad, W = \mathbb{C}^w$$

$G = GL(V)$
$N = \text{Hom}(W, V) \curvearrowleft G \lhd \widetilde{G} = G \times GL(W)$

$M = N \oplus N^*$

$$\begin{array}{c} V \\ b \downarrow \uparrow a \\ W \end{array}$$

$\mu = ab$
$\in \mathfrak{gl}(V) \cong \mathfrak{gl}(V)^*$

$$\mathcal{M}_H = \mu^{-1}(0) /\!/ GL(V) = \text{Spec}\left(\mathbb{C}[\mu^{-1}(0)]^G\right)$$

$X := ba \in \text{End}(W) \quad (G\text{-invariant})$

$$X^2 = ba\underbrace{ba}_{0} a = 0$$

$$_{\ni} [a, b] \mod GL(V) \mapsto ba$$

$\underline{\text{Fact}}.$ $\mathcal{M}_H \cong \{ X \in \text{End}(W) \mid X^2 = 0, \text{rk} X \leq v \}$

If $v \geq w/2$, $\text{rk} X \leq v$ is
automatically satisfied

$$\left(X \sim (22 \cdots 1 \cdots 1) \right)$$

$$\zeta \in \text{Hom}\,(G, \mathbb{C}^{\times})$$
$$\|\ \text{det}$$

$\mathbb{Z} \quad b : \text{injective}$

$$\text{GIT} \qquad \mu^{-1}(0) \underset{\zeta}{/\!\!/} G \;=\; \mu^{-1}(0)^{\overbrace{\zeta-st}} / G$$

$$\mathcal{M}_H^{\zeta} \;\cong\; T^* \text{Gr}(v, w)$$
$$\overset{\omega}{}$$
$$[a, b] \longmapsto (\text{Im}\,b,\; X = ba)$$
$$\underset{\cap}{}$$
$$W \qquad \cdot X(\text{Im}\,b) = 0$$
$$\cdot \,\text{Im}\,X \subset \text{Im}\,b$$

Grassmannian
of v-dim.
subsp. in $W = \mathbb{C}^w$

$$\text{If} \quad v > w \quad \Rightarrow \quad \mathcal{M}_H^{\zeta} = \phi$$

$$v = w \qquad \mathcal{M}_H^{\zeta} = pt \longrightarrow \mathcal{M}_H$$
$$\|$$
$$\{X \mid X^2 = 0\}$$

$\underline{\text{Fact.}}$ $\text{If} \quad v \leq w/2 \quad \Rightarrow \quad \mathcal{M}_H^{\zeta} \rightarrow \mathcal{M}_H$
$\qquad\qquad\qquad\qquad\qquad$ resol. of singularities

$$\zeta_{\mathbb{C}} = d\zeta \;\in\; \text{Hom}_{\text{Lie alg}}(\mathfrak{g}, \mathbb{C})$$
$$\diagdown \ \text{scalar matrix}$$

$$\mathcal{M}_H^{\zeta_{\mathbb{C}}} = \mu^{-1}(\zeta_{\mathbb{C}}) / G \;\cong\; \begin{array}{l}\text{coadjoint}\\ \text{orbit through}\end{array} \begin{bmatrix} 0 & & & \overbrace{w-v} \\ & \ddots & & 0_v \\ & & 0 & \\ 0 & & & \zeta_{\mathbb{C}} \ddots \\ & & & \quad \zeta_{\mathbb{C}} \end{bmatrix}$$

§ Convolution algebra

X : finite set

$\mathcal{F}i(X \times X) = \mathbb{C}$-valued fcts on $X \times X$

$\downarrow$

$K, K' \qquad K * K'(x,z) \overset{\text{def}}{=} \sum_{y \in X} K(x,y) K'(y,z)$

$\mathcal{F}i(X \times X)$ is an associative algebra

$\| \,$

$\text{Mat}_{n \times n}(\mathbb{C}) \qquad n = |X|$

G finite group $\curvearrowright X$

$G \curvearrowright X \times X$ diagonal action

$\mathcal{F}i(X \times X)^G \qquad G$-inv. functions on $X \times X$

$\llcorner$ subalgebra

$G = GL_n(\mathbb{F}_q) \supset B = $ upper triangular part

$X = G/B$ flag variety over $\mathbb{F}_q$

$G \curvearrowright X \times X \qquad$
$\begin{array}{ccc} G\text{-orbits} & \overset{1:1}{\longleftrightarrow} & B\text{-orbits} \\ \text{in } X \times X & & \text{in } X \end{array}$

$\longleftrightarrow \; S_n :$ symmetric grp

T_6 (Iwahori)

$H(X \times X)^G$ generators T_i $i = 1, \ldots, n-1$

relations $(T_i + 1)(T_i - g) = 0$

$$T_i T_j = T_j T_i \quad \text{if} \quad |i-j| > 1$$

$$T_i \, T_{i+1} \, T_i = T_{i+1} \, T_i \, T_{i+1}$$

deformation of $\mathbb{C}[G_n]$

$\uparrow$ Iwahori - Hecke algebra

For the definition of Coulomb branches
we use convolution algebra on
_equivariant Borel - Moore homology
group_.

$\Big($ cf (non-equiv.) Borel - Moore homology
Fulton : Young tableaux

M : smooth and cpt , oriented

$G \curvearrowright$

$$H_*^G(M) \cong H_G^{\dim M - *}(M) \quad (\text{Poincaré duality})$$

M : non cpt $\diagdown$ replaced homology of
by locally finite
chains

functorial under proper morphisms

$M \xrightarrow{\pi} X$ proper

smooth

$\overset{\mathfrak{I} \quad G \quad \mathfrak{I}}{}$ - equivariant

$$Z = M \underset{X}{\times} M = \{(x_1, x_2) \in M \times M \mid \pi(x_1) = \pi(x_2)\}$$

$\curvearrowright$ closed

$M \times M$

$$M \times M \times M \xrightarrow{P_{ij}} M \times M$$

$(i,j) = (1,2), (2,3)$
$(1,3)$

$$P_{12}^{-1}(Z) \cap P_{23}^{-1}(Z) = \{(x_1, x_2, x_3) \mid \begin{matrix} \pi(x_1) \\ \shortparallel \\ \pi(x_2) \\ \shortparallel \\ \pi(x_3) \end{matrix} \}$$

$\downarrow P_{13}$ proper

Z

$H_*^G(Z)$ convolution product

$\Downarrow$

K, K' $P_{13*} \left(\underbrace{P_{12}^*(K)}_{\substack{\uparrow \\ H_*^G(P_{12}^{-1}Z)}} \cap \underset{M \times M \times M}{\underbrace{P_{23}^*(K')}_{\substack{\uparrow \\ H_*^G(M \times Z)}}} \right) =: K * K'$

$\qquad\qquad\qquad \underbrace{\phantom{H_*^G(P_{12}^{-1}Z)}}_{Z \times M}$

$H_*^G(Z)$ with $*$ is an associative algebra.
(convolution algebra)

usually non commutative.

$H_G^*(M)$ with $\cup$

$$M \longrightarrow Z = M \times_X M$$

diagonal
embedding

$$H_*^G(M) \longrightarrow H_*^G(Z)$$

$\cup$

$*$

alg. from.

Example
Fix w

$$\coprod_{0 \leq v \leq w} T^* Gr(v, w) \longrightarrow \{X \in End(w) \mid X^2 = 0\}$$

$\|$

$\mathcal{M}_H^s$

$\|$

$\mathcal{M}_H$

$\underbrace{\qquad\qquad}_{\|}$

M

$\underbrace{\qquad\qquad}_{\|}$

X

$$Z = M \times_X M$$

$$= \coprod_{v_1, v_2} T^* Gr(v_1, w) \times T^* Gr(v_2, w)$$

$GL(w) \times \mathbb{C}^\times$

$\underbrace{\qquad}$

scaling in
cotangent
direction
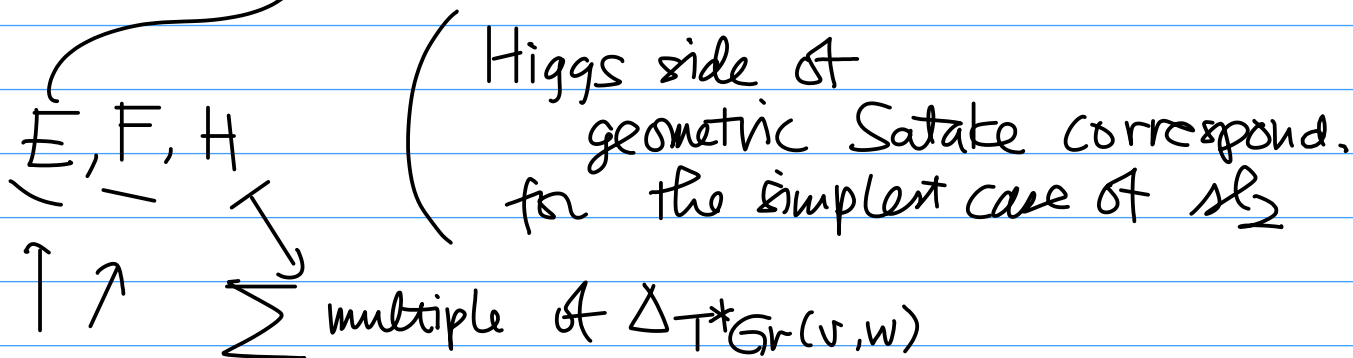
$$H_{top}(Z) = \bigoplus_{v_1, v_2} H_{top}(\qquad)$$

lagrangian
subvar.
in $T^* Gr(v_1, w) \times T^* Gr(v_2, w)$

Th [Ginzburg]

$$U(sl_2) \longrightarrow H_{top}(Z)$$

$\exists$ alg. hom.

E, F, H $\Bigg($ Higgs side of geometric Satake correspond. for the simplest case of sl_2

$\uparrow \nearrow$ $\sum$ multiple of $\Delta_{T^*Gr(v,w)}$

correspondences

$$T^*Gr(v, w) \times T^*Gr(v+1, w)$$

$\cup$ smooth subv.

$$\{(S_1, X, S_2) \mid S_1 \subset S_2\}$$

$\underset{W}{\wedge}$ $\underset{W}{\wedge}$

$$F \mapsto \sum_v \pm [\text{fund. class of } \rceil]$$

$$E \mapsto \sum \pm [\text{exchange } v \leftrightarrow v+1]$$

$\Bigg($ $H_*^{G \times \mathbb{C}^*}(Z) \longleftarrow$ Yangian

$K^{G \times \mathbb{C}^*}(Z) \longleftarrow U_q(\text{affine } sl_2)$