All about deformations of cyclic quotient surface singularities

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Correspondence between various descriptions of deformations of cyclic quotient surface singularities via the semi-stable minimal model program

What is ··· a cyclic quotient surface singularity?

$$\frac{1}{n}(1, a) = \mathbb{C}^2/\mu_n \text{ via } \zeta \cdot (x, y) = (\zeta x, \zeta^a y)$$

- Hirzubruch-Jung continued fractions

$$\frac{n}{a} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_r}}}, \quad b_i \geq 2$$

- Minimal resolutions

$$\frac{1}{19}(1,7)$$

- Hirzubruch-Jung continued fractions

$$\frac{19}{7} = 3 - \frac{1}{4 - \frac{1}{2}}$$

- Minimal resolutions

Three descriptions of deformations

- **Equations**
- **P-resolutions**
- **Picture deformations**

Equations

Equations and their deformations

[Christophersen 1991], [Stevens 1991]

$$z_{i+1}z_{i-1} = z_i^{a_i} + t \cdot z_i^{k_i}$$

$\frac{1}{n}(1,a)$

- Dual Hirzebruch-Jung continued fractions

$$\frac{n}{n-a} = [a_1, \cdots, a_e]$$

- $K(n/n-a) = \{ (k_1, \cdots, k_e) \mid [k_1, \cdots, k_e] = 0 \ + \ \text{—} \}$

$k_i \geq 1 \quad + \quad k_i \leq a_i$

$[1, 1] \quad [2\ 1\ 2] \quad [3\ 1\ 2\ 3]$

Example

$$\frac{1}{19}(1,7)$$

- Dual Hirzebruch-Jung continued fractions

$$\frac{19}{19-7} = [2,3,2,3]$$

- $K(19/19-7)$ $= \{(1,2,2,1), (1,3,1,2), (2,2,1,3)\}$

Irreducible components

P-resolutions

What is ⋯ Kollár Conjecture

[Kollár–Shepherd-Barron 1988]

Let $(X, 0)$ be a rational surface singularity and let $\mathcal{X}$ be the total space of a one-parameter smoothing of $(X, 0)$. Then the canonical algebra

$$\sum_{n=0}^{\infty} \mathcal{O}_{\mathcal{X}}(nK_{\mathcal{X}})$$

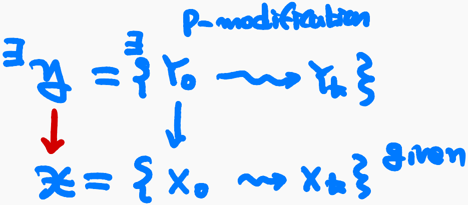
is a finitely generated $\mathcal{O}_{\mathcal{X}}$-algebra.

$$\mathcal{X} = \{ X_0 \rightsquigarrow X_t \}$$

What is ··· Kollár Conjecture (rough)

[Kollár–Shepherd-Barron 1988]

For any rational surface singularity X_0, there is a one-to-one correspondence between irreducible components of the reduced miniversal deformation space of X_0 and P-modifications of X_0.



Kollár Conjecture holds for ⋯

- smoothings over the Artin component
  - rational double points, rational triple points
- quotient surface singularities; [Kollár-Shepherd-Barron 1988]
- quotients of simple elliptic and cusp singularities; [Kollár-Shepherd-Barron 1988]
- rational quadruple points; [Stevens 1991]

Kollár Conjecture holds for quotient surface singularities [Kollár–Shepherd-Barron 1988]

For any quotient surface singularity X_0, there is a one-to-one correspondence between irreducible components of the reduced miniversal deformation space of X_0 and P-resolutions of X_0.

What is ⋯ a P-resolution?

A partial resolution $f\colon Y \to X$ with T-singularities and the ample K_Y relative to f

What is ⋯ a T-singularity?

$$\frac{1}{dn^2}(1, dna-1) \text{ (or RDP)}$$

- [4]
- [2 5] $\quad 2 - \frac{1}{5} = \frac{9}{5}$

$= \frac{1}{3^2}(1,5)$

- [5, 2]

- one-parameter $\mathbb{Q}$-Gorenstein smoothing

- Generating algorithm

$d=1 \qquad \frac{1}{n^2}(1, na-1)$

[4] $\qquad$ $[b_1, \cdots, b_r]$: T-sing

$\frac{1}{4}(1,1) \qquad \Rightarrow \qquad [2, b_1, \cdots, b_{r-1}, b_r+1]$

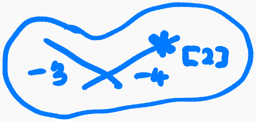
$[b_1+1, b_2, \cdots, b_r, 2]$

Example

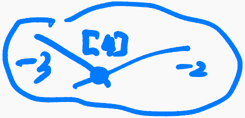
$$\frac{1}{19}(1,7)$$

[3, 4, 2]

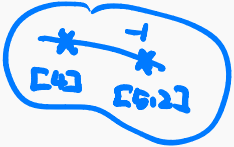
- 3 − 4 − [2]



- 3 − [4] − 2



- [4] − 1 − [5, 2]



Picture deformations

What is ⋯ a sandwiched surface singularity?

[Hironaka, Spivakovsky, ⋯]

A normal surface singularity that admits a birational morphism to $\mathbb{C}^2$

$$(V, E) \to (X, 0) \longrightarrow (\mathbb{C}^2, 0)$$

Digression: Why ⋯ a sandwiched surface singularity?

Digression: Why ··· a sandwiched surface singularity?

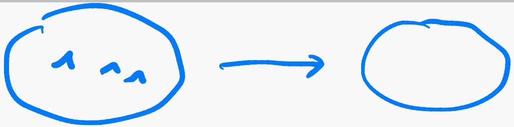
J. Nash asked to H. Hironaka in the early sixties:

> “Does a finite succession of Nash transformations resolve the singularities of algebraic varieties?”

Digression: Why ··· a sandwiched surface singularity?

H. Hironaka proved in 1983 that:

> After a finite succession of normalized Nash transformations we obtain a surface which birationally dominates a non-singular surface.



Digression: Why ⋯ a sandwiched surface singularity?

M. Spivakovsky proved in 1990 that:

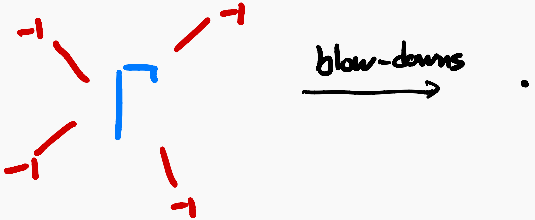
> “Sandwiched singularities are resolved by a finite sequence of normalized Nash transformations.”

What is ⋯ a sandwiched surface singularity?

Sandwiched singularities are rational and characterized by their dual resolution graphs.

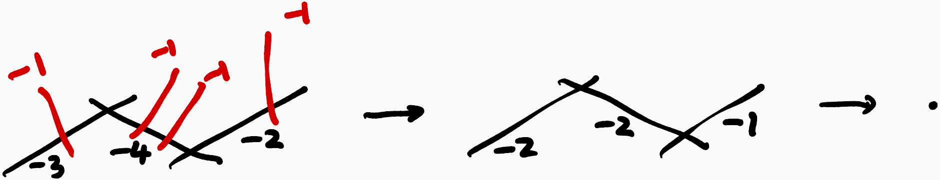
What is ⋯ a sandwiched graph?

A dual resolution graph that blows down to a smooth point by adding (-1)-vertices to some of vertices.



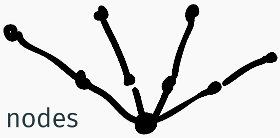
Example

$$\frac{1}{19}(1,7)$$



Examples of sandwiched surface singularities

- Cyclic quotient surface singularities
- Weighted homogeneous surface singularities with “big” central nodes
- Rational surface singularities with the reduced fundamental cycles



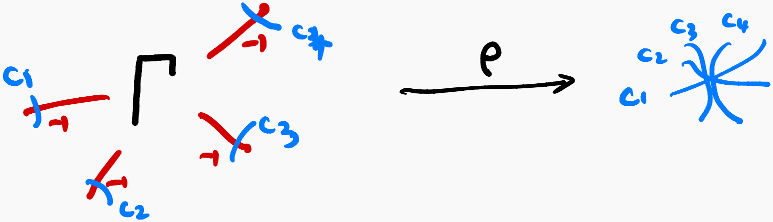
Deformations of sandwiched surface singularities

[de Jong-van Straten 1998]

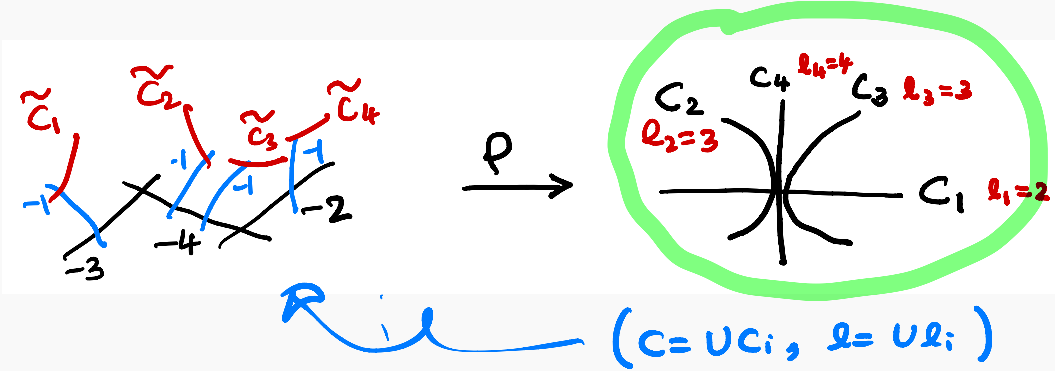
All the one-parameter deformations of a sandwiched surface singularity are obtained by one-parameter deformations of the corresponding decorated curve. Moreover, all the smoothings of the singularity are provided by picture deformations of the decorated curves.

What is ⋯ a decorated curve?

A decorated curve (C, l) is a union C of small pieces of curves C_i intersecting each sandwiched (-1)-curves F_i together with integers l_i recording the number of blow-ups occurring on the strict transformations of C_i starting from the smooth point.



Example



$\tilde{C}_1$
$\tilde{C}_2$
$\tilde{C}_3$
$\tilde{C}_4$
-1
-1
-1
-1
-1
-3
-4
-2
ρ
C_2
$\ell_2=3$
C_4
$\ell_4=4$
C_3
$\ell_3=3$
C_1
$\ell_1=2$
$(C=\cup C_i, \ \ell=\cup \ell_i)$

What is ⋯ a picture deformation?

A picture deformation of a decorated curve (C, l) over S consists of

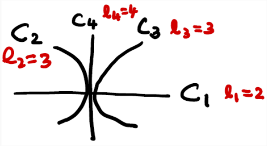
(a) a δ-constant deformation $C_S \to S$ of C,

(b) a flat deformation $l_S \subset \widetilde{C}_S = C_S \times S$ of the scheme l such that

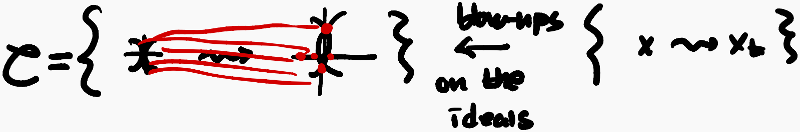
(c) $m_S \subset l_S$, where the relative total multiplicity scheme m_S of $\widetilde{C}_S \to C$ is defined as the closure $\bigcup_{s \in S \setminus 0} m(C_s)$.

(d) for generic $s \in S \setminus 0$ the divisor l_s on $\widetilde{C}_s$ is reduced.

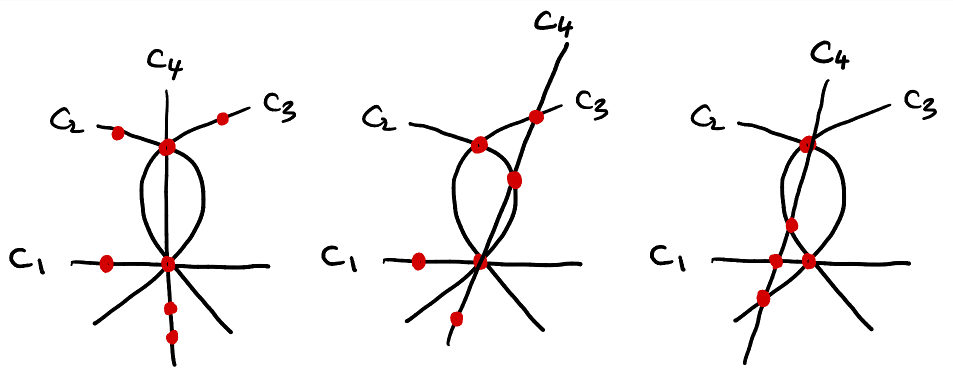
Example



C2
ℓ2=3
C4
ℓ4=4
C3
ℓ3=3
C1
ℓ1=2

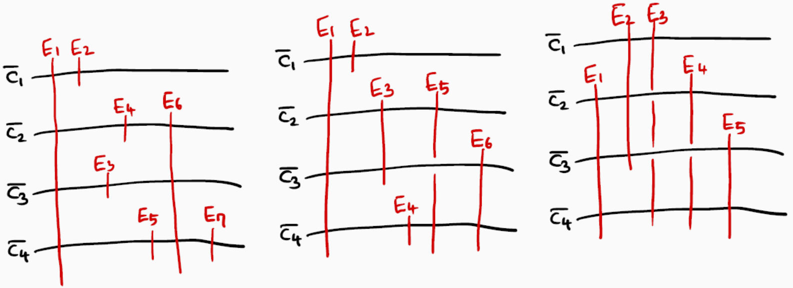


C = {
}
blow-ups
on the ideals
{ x ⇝ x_t }



C4
C3
C2
C1
C4
C3
C2
C1
C4
C3
C2
C1

Example



$$\begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\begin{array}{c} E_1 \; E_2 \qquad\qquad\qquad E_7 \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Correspondence between the three descriptions

- **Equations**
- **P-resolutions**
- **Picture deformations**

Example

$$\frac{1}{19}(1,7)$$

Def(X_0)의 irreducible components

- **Equations**
  - $(1,2,2,1)$, $(1,3,1,2)$, $(2,2,1,3)$
- **P-resolutions**
  - $3-4-[2]$, $3-[4]-2$, $[4]-1-[5,2]$
- **Picture deformations**
  - $$\begin{bmatrix}1&1&0&0&0&0&0\\1&0&0&1&0&1&0\\1&0&1&0&0&1&0\\1&0&0&0&1&1&1\end{bmatrix}, \begin{bmatrix}1&1&0&0&0&0\\1&0&1&0&1&0\\1&0&1&0&0&1\\1&0&0&1&1&1\end{bmatrix}, \begin{bmatrix}0&1&1&0&0\\1&1&0&1&0\\1&1&0&0&1\\1&0&1&1&1\end{bmatrix}$$

Correspondence using topology

The topological method in [Némethi-Popescu-Pampu 2010]

Equations $\to$ Symplectic fillings $\leftarrow$ Picture deformations

What is ··· a symplectic filling?

A symplectic 4-manifold with boundary that is compatible with the Milnor fillable contact structure of the boundary

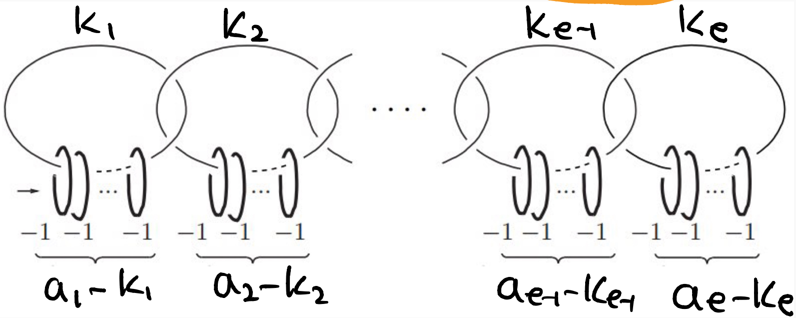
Lisca's classification of symplectic fillings of $L(n, a)$

[Lisca 2008]

$W_{n,a}(\underline{k})$

$\underline{k} \in K(n/n-a)$

$\frac{1}{n}(n,a)$ = $L(n,a)$

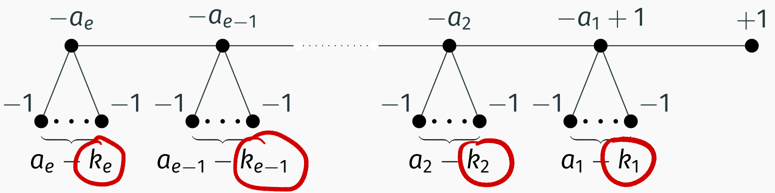


Lisca's classification of symplectic fillings of $L(n, a)$ [Lisca 2008]

$W_{n,a}(\underline{k})$

$[a_1, \cdots, a_e] = \frac{n}{n-a}$ ✓

$=$ complement of



$\subset \mathbb{CP}^2 \# N\overline{\mathbb{CP}^2}$

$\longleftrightarrow \quad (k_1, \cdots, k_e) \in K(n/n-a)$

Lucky!

[Némethi-Popescu-Pampu 2010]

$$W_{n,a}(\underline{k})^* \ncong W_{n,a}(\underline{k}')^* \quad \text{iff} \quad \underline{k} \neq \underline{k}'$$

$$\cong \quad \iff \quad \underline{k} = \underline{k}'$$

Correspondence using topology

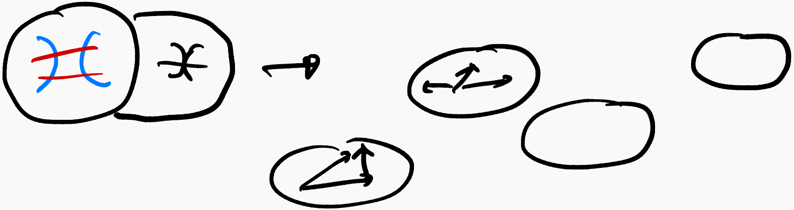
[Némethi-Popescu-Pampu 2010]

$X_0 \rightsquigarrow X_t \cong$ "$\mathbb{CP}^2 \# N \overline{\mathbb{CP}^2} - *$" $\to \underline{k}$

Equations → Milnor fibers as complements → Symplectic fillings $W_{n,a}(\underline{k})$

$\cong$ $\cong$

Picture deformations → Milnor fibers as complements → Symplectic fillings $W_{n,a}(\underline{k}')$



Correspondence using MMP

Correspondence using MMP

From P-resolutions to Complements

From P-resolutions to Complements

[H. Park-J. Park-D. Shin-G. Urzúa 2018]

Step 1. Compactifying P-resolutions

$$\mathcal{Z} = \left\{ \boxed{Y_0 - 1 - a_e - \cdots - a_2 - a_1 - 1 - (+1)} \rightsquigarrow \boxed{Y_t - a_e - \cdots - a_2 - a_1 - 1 - (+1)} \right\}$$

$$\mathcal{Y} = \{ Y_0 \rightsquigarrow Y_t \} \quad \subset \quad \mathcal{Z} = \{ \bar{Y}_0 \rightsquigarrow \bar{Y}_t \}$$

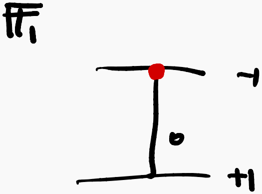
$$\downarrow \qquad \downarrow$$

$$\mathcal{X} = \{ X_0 \rightsquigarrow X_t \} \qquad \qquad \leftarrow H^2(Z, T_Z) = 0$$

Example

$$\frac{1}{19}(1,7)$$

$$\{3 - [4] - 2 - 1 - 3 - 2 - 3 - 2 \quad \rightsquigarrow \quad 3 - 2 - 3 - 2\}$$



Step 2. Applying the semi-stable MMP

By applying only Iitaka-Kodaira divisorial contractions and usual flips, one can run MMP to $\mathcal{Z} \to \Delta$ until we obtain a deformation $\mathcal{Z}' \to \Delta$ whose central fiber Z_0' is smooth.

$$\mathcal{Z} = \{ Z_0 \rightsquigarrow Z_t \} \overset{\text{MMP}}{\dashrightarrow} \mathcal{Z}^+ = \{ Z_0^+ \rightsquigarrow Z_t^+ \}$$

Example

E_1 E_1

$\{3 - [4] - 2 - 1 - 3 - 2 - 3 - 2 \quad \rightsquigarrow \quad 3 - 2 - 3 - 2\}$

↓ divisorial contraction

c^-

$\{3 - [4] - 1 - x - 2 - 2 - 3 - 2 \quad \rightsquigarrow \quad 2 - 2 - 3 - 2\}$

↓ flip

$\{3 - 3 - x - x - 1 - 2 - 3 - 2 \quad \rightsquigarrow \quad 2 - 2 - 3 - 2\} = Z^+$

c^+

From P-resolutions to Complements

[H. Park-J. Park-D. Shin-G. Urzúa 2018]

Step 3. Identifying Milnor fibers

- The central fiber Z'_0 is diffeomorphic to a general fiber Z'_t.
- By comparing Z'_0 and Z'_t, one can get the data of positions of (-1)-curves in Z'_t.
- One can get the data of (-1)-curves on Z_t by tracking the blow-downs $Z_t \to Z'_t$ given by flips and divisorial contractions.

Example

$$\{3 - [4] - 2 - 1 - 3 - 2 - 3 - 2 \quad \rightsquigarrow \quad 3 - 2 - 3 - 2\}$$

$$\{3 - [4] - 1 - x - 2 - 2 - 3 - 2 \quad \rightsquigarrow \quad 2 - 2 - 3 - 2\}$$

$$\{3 - 3 - x - x - 1 - 2 - 3 - 2 \quad \rightsquigarrow \quad 2 - 2 - 3 - 2\}$$

$$\{3 - 2 - x - x - x - 1 - 3 - 2 \quad \rightsquigarrow \quad 2 - 1 - 3 - 2\}$$

$$\{3 - 1 - x - x - x - x - 2 - 2 \quad \rightsquigarrow \quad 1 - x - 2 - 2\}$$

$$\{2 - x - x - x - x - x - 1 - 2 \quad \rightsquigarrow \quad x - x - 1 - 2\}$$

$$\{1 - x - x - x - x - x - x - 1 \quad \rightsquigarrow \quad x - x - x - 1\}$$

Correspondence using MMP

From P-resolutions to Picture deformations

From P-resolutions to Picture deformations

[H. Park-D. Shin 2021(?)]

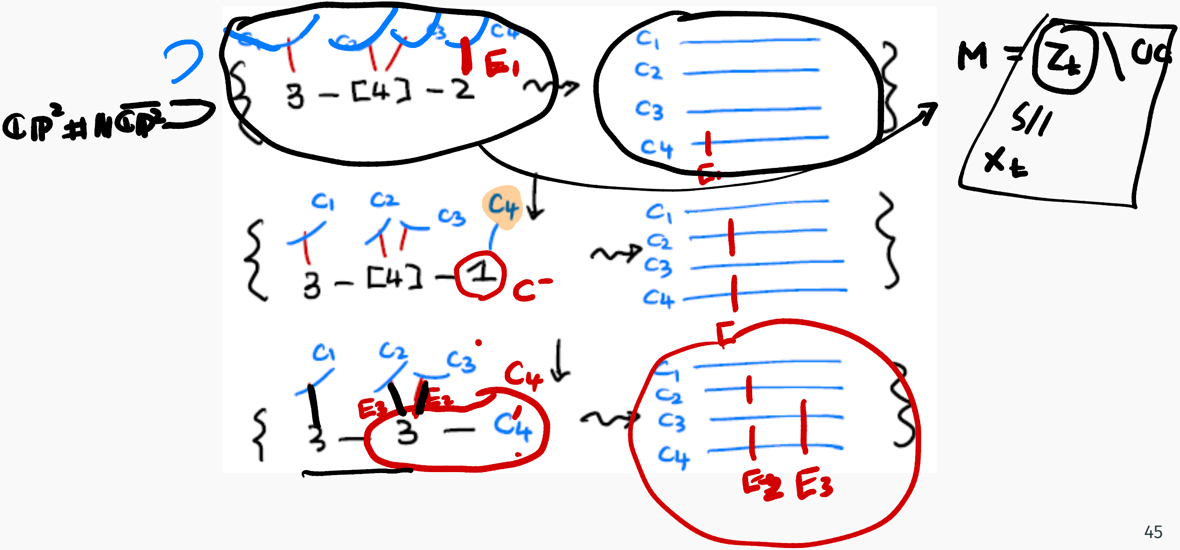
For any sandwiched surface singularities, one can obtain picture deformations from P-resolutions via the semi-stable MMP.

From P-resolutions to Picture deformations

[H. Park-D. Shin 2021(?)]

- Step 1. Compactifying P-resolutions
- Step 2. Applying the semi-stable MMP
- Step 3. Identifying Picture deformations

Example

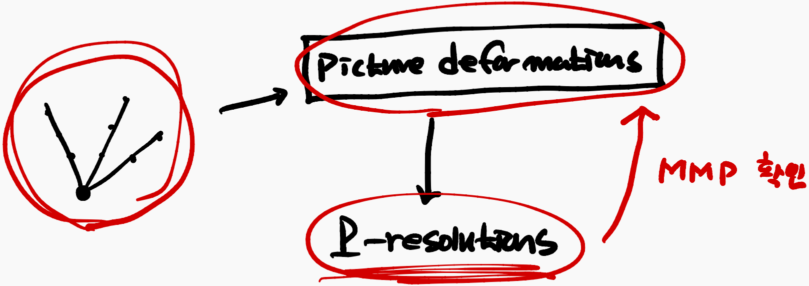


$\mathbb{C}P^2 \# N\overline{\mathbb{C}P^2}$
E_1
$3 - [4] - 2$
C_1
C_2
C_3
C_4
$3 - [4] - 1$
C^-
E_2
E_3
$M = (Z_t) \setminus$
$S/\!/$
X_t

Applications

Kollár Conjecture for Weighted homogeneous surface singularities [J. Jeon, D. Shin ????]

Try to prove Kollár Conjecture for weighted homogeneous surface singularities with “big” central nodes.



감사합니다!

$$(1,2,2,1) = 3-4-[2] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$(1,3,1,2) = 3-[4]-2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

$$(2,2,1,3) = [4]-1-[5,2] = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$