Higher secant varieties and the identifiability

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"A matryoshka structure of higher secant varieties and the generalized Bronowski's conjecture" (arXiv:2103.02412 [math.AG])

- / an algebraically closed field of characteristic zero.
- $X \subseteq \mathbb{P}^r$: a nondegenerate projective variety.
  1. irreducible and reduced
  2. $\langle X \rangle = \mathbb{P}^r$
- $q \in \mathbb{Z}$: a positive integer.

Definition

The *q-secant variety* $S^q(X)$ to $X \subseteq \mathbb{P}^r$ is defined by

$$S^q(X) = \overline{\bigcup \langle z_1, \ldots, z_q \rangle} \subseteq \mathbb{P}^r,$$

where the $\langle z_1, \ldots, z_q \rangle$ are q-secant $(q-1)$-planes to X with $z_i \in X$.

We define the *abstract q-secant variety* $\widetilde{S}^q(X)$ to $X \subseteq \mathbb{P}^r$ by

$$\widetilde{S}^q(X) = \overline{\{(\zeta, w) : \zeta \in U, w \in \langle \zeta \rangle\}} \subseteq X^{(q)} \times \mathbb{P}^r,$$

where $U \subseteq X^{(q)}$ consists of sets (0-cycles) of q linearly independent points of X. So we have a surjective morphism

$$\widetilde{S}^q(X) \to S^q(X).$$

Suppose that $\dim S^q(X) = q \dim X + q - 1$ (cf. *nondefective*), hence the map $\widetilde{S}^q(X) \to S^q(X)$ is generically finite. For a general point $w \in S^q(X)$ we have

$$\deg(\widetilde{S}^q(X) \to S^q(X)) = \#\{\zeta \in U : \langle \zeta \rangle \ni w \text{ in } \mathbb{P}^r\}.$$

Definition

The projective variety $X \subseteq \mathbb{P}^r$ is *q-identifiable* if the map $\widetilde{S}^q(X) \to S^q(X)$ is birational.

On the other hand,

Definition

Let $\Lambda \subsetneq \mathbb{P}^r$ be a λ-plane $x_{\lambda+1} = \ldots = x_r = 0$. Then the *projection* of $X \subseteq \mathbb{P}^r$ from Λ is a map $\pi_\Lambda : X \setminus \Lambda \to \mathbb{P}^{r-\lambda-1}$ defined by

$$(x_0 : \cdots : x_\lambda : x_{\lambda+1} : \cdots : x_r) \mapsto (x_{\lambda+1} : \cdots : x_r).$$

Write $X_\Lambda = \overline{\pi_\Lambda(X \setminus \Lambda)} \subseteq \mathbb{P}^{r-\lambda-1}$.

1. Taking $\Lambda = \{z\} \subseteq X$ we have an *inner projection* of X.
   $\rightsquigarrow$ a *general s-inner projection*
2. Taking $\Lambda = \mathbb{T}_z X \subseteq \mathbb{P}^r$ $(z \in X)$ we have a *tangential projection* of X.
   $\rightsquigarrow$ a *general s-tangential projection*

For any λ-plane $\Lambda \subsetneq \mathbb{P}^r$ we obtain

$$S^q(X)_\Lambda = S^q(X_\Lambda) \subseteq \mathbb{P}^{r-\lambda-1}.$$

Note that if $S^q(X) \neq \mathbb{P}^r$, then $S^j(X) \subseteq \mathrm{Sing} S^q(X)$ for every $1 \leq j < q$.

Theorem (Ciliberto–Russo, 2006)

Let $\Lambda = \mathbb{T}_w S^j(X)$ be a general projective tangent space to $S^j(X)$. Then for the projective tangent cone $\mathbb{T}C_w S^q(X) \subseteq \mathbb{P}^r$ we have

$$\mathrm{Cone}_\Lambda(S^{q-j}(X_\Lambda)) \subseteq \mathbb{T}C_w S^q(X),$$

and it is moreover an irreducible component.

- $\mathrm{Cone}_{\mathbb{T}_z X}(S^{q-1}(X_{\mathbb{T}_z X}))$ when a point $z \in S^1(X) = X$ is general.

Definition

The projective variety $X \subset \mathbb{P}^r$ has *one apparent double point* if the map $\widetilde{S}^2(X) \to S^2(X)$ is birational with $S^2(X) = \mathbb{P}^r$.

- : the simplest case of *nondefective* varieties $X^n \subset \mathbb{P}^{2n+1}$.

By [Severi, 1901], [Russo, 2000] (cf. [Ciliberto–Mella–Russo, 2004]) and [Ciliberto–Russo, 2011], we have:

Proposition (as above)

If $X \subset \mathbb{P}^5$ is a surface with one apparent double point, then it is

1. *a smooth rational normal scroll;*
2. *a (weak) del Pezzo surface; or*
3. *a Verra surface.*

This is about

$$\{\text{sums of } q \text{ linear forms to the } d\text{-th power}\} \subseteq \{\text{homogeneous } d\text{-forms}\}.$$

Let $e = \operatorname{codim} S^q(\nu_d(\mathbb{P}^n))$ for the d-Veronese variety $\nu_d(\mathbb{P}^n)$. Consider a polynomial ring in $n+1$ variables.

- f: a general homogeneous polynomial of degree d.
- L_i: e general linear forms.

Then there exist some q linear forms ℓ_i such that

$$f = L_1^d + \cdots + L_e^d + \ell_1^d + \cdots + \ell_q^d.$$

Question

Is such an expression $f = \sum_i L_i^d + \sum_j \ell_j^d$ unique up to symmetry?

$$\nu_d(\mathbb{P}^n) = \{[\ell^d] : \ell \text{ is a linear form}\}$$

Question

In other words, is a general e-inner projection of $\nu_d(\mathbb{P}^n)$ q-identifiable?

- In 1933, Bronowski studied this problem, and made a conclusion.
- "We called the above claim *Bronowski claim* since the proof proposed by Bronowski is obscure, as far as I know, to all modern algebraic geometers who read it." from [Russo, 2003]
  $\rightsquigarrow$ the *generalized Bronowski's conjecture* [Ciliberto–Russo, 2006]

We have

$$\deg X \geq e+1, \quad e := \mathrm{codim} X,$$

and if equality holds, then by definition, X is a *variety of minimal degree*.

[Ciliberto–Russo, 2006] tells us that

$$\deg S^q(X) \geq \binom{e+q}{q}, \quad e := \mathrm{codim} S^q(X).$$

Definition

One says that $S^q(X)$ is a *q-secant variety of minimal degree* if the equality above holds.

According to [Ciliberto–Russo, 2006], the following hold:

Theorem (Ciliberto–Russo)

If $S^q(X)$ has minimal degree, then for a general point $z \in X$

1. *$S^q(X_z)$ has minimal degree; and*
2. *$S^{q-1}(X_{\mathbb{T}_z X})$ has minimal degree.*

Theorem (Ciliberto–Russo)

Suppose that $S^q(X)$ has minimal degree of codimension e. Let X_Λ be a general e-inner projection of X. Then

$$X \text{ is } q\text{-identifiable} \iff X_\Lambda \text{ is } q\text{-identifiable.}$$

The *generalized Bronowski's conjecture* says:

Conjecture (Ciliberto–Russo, 2006)

The following are equivalent:

1. *X is q-identifiable and $S^q(X)$ has minimal degree.*
2. *A general $(q-1)$-tangential projection of X is birational onto a variety of minimal degree.*

- (1) $\Rightarrow$ (2) ([Ciliberto–Mella–Russo, 2004] & [Ciliberto–Russo, 2006]).

Assume that it is true.

1. If a general tangential projection of $X^n \subset \mathbb{P}^{2n+1}$ is birational onto $\mathbb{P}^n$, then X is a variety with one apparent double point.
2. If the d-Veronese variety $\nu_d(\mathbb{P}^n)$ satisfies (2), then the expression $f = \sum_i L_i^d + \sum_j \ell_j^d$ above is unique up to symmetry.

Main theorem

Theorem (C.–Kwak)

Let $e = \operatorname{codim} S^q(X)$*, and* $z \in X$ *stand for a general point. Then the following hold:*

1. *When* $e \geq 2$ *and* $q \geq 1$*,*
$$S^q(X) \text{ has minimal degree} \iff S^q(X_z) \text{ has minimal degree.}$$
2. *When* $e \geq 1$ *and* $q \geq 2$*,*
$$S^q(X) \text{ has minimal degree} \iff S^{q-1}(X_{\mathbb{T}_z X}) \text{ has minimal degree.}$$

- : the algebraic part of the generalized Bronowski's conjecture.

Corollary (C.–Kwak)

The generalized Bronowski's conjecture holds for the following (X, q)*:*

1. X *is a curve with* $q \geq 1$*; and*
2. X *satisfies that* $\dim S^{q+1}(X) = (q+1)\dim X + q$*.*

- By [Ciliberto–Russo, 2006], the generalized Bronowski's conjecture is true for smooth surfaces.

Theorem (C.–Kwak (simplified), cf. [Han-Kwak, 2015])

The q-secant variety $S^q(X)$ *has minimal degree of codimension* $e \geq 1$ *if and only if the minimal free resolution of* $S_{S^q(X)}$ *is of the form*

$$S \leftarrow S^{\beta_1}(-q-1) \leftarrow S^{\beta_2}(-q-2) \leftarrow \cdots \leftarrow S^{\beta_e}(-q-e) \leftarrow 0$$

with $\beta_p := \binom{p+q-1}{q}\binom{e+q}{p+q}$.

Plus

Varieties with one apparent double point $\rightsquigarrow$

Definition

The projective variety $X^n \subset \mathbb{P}^{2n+1}$ has *two apparent double points* if

$$S^2(X) = \mathbb{P}^{2n+1} \quad \text{and} \quad \deg(\widetilde{S}^2(X) \to S^2(X)) = 2.$$

The uniqueness for Waring's problem on homogeneous forms $\rightsquigarrow$

Question

When is the number of the expressions $f = \sum_i L_i^d + \sum_j \ell_j^d$ above two?

Plus

Higher secant varieties of almost minimal degree

Proposition

If $S^q(X)$ does not have minimal degree, then

$$\deg S^q(X) \geq \binom{e+q}{q} + \binom{e+q-1}{q-1}, \quad e := \mathrm{codim} S^q(X).$$

$\rightsquigarrow$ *almost minimal degree*

Proposition

Suppose that $S^q(X)$ is of almost minimal degree with codimension $e \geq 1$. Then

$$\deg(\widetilde{S}^q(X_\Lambda) \to S^q(X_\Lambda)) = 2 \Longleftrightarrow X \text{ is } q\text{-identifiable}$$

for a general e-inner projection X_Λ of X.

Proposition

If $S^q(X)$ has almost minimal degree of codimension e, then the sectional genus satisfies

$$\pi(S^q(X)) \leq (q-1)\deg S^q(X) + 1.$$

$\rightsquigarrow$ a *del Pezzo q-secant variety*

Theorem (C.–Kwak)

Let $e = \mathrm{codim} S^q(X)$, and $z \in X$ stand for a general point. Then the following hold:

1. *When $e \geq 3$ and $q \geq 1$,*
$$S^q(X) \text{ is del Pezzo} \iff S^q(X_z) \text{ is del Pezzo.}$$
2. *When $e \geq 2$ and $q \geq 3$,*
$$S^q(X) \text{ is del Pezzo} \iff S^{q-1}(X_{\mathbb{T}_z X}) \text{ is del Pezzo.}$$

Theorem (C.–Kwak (simplified), cf. [Han-Kwak, 2015])

The q-secant variety $S^q(X)$ is del Pezzo of codimension $e \geq 2$ if and only if the minimal free resolution of $S_{S^q(X)}$ is of the form

$$S \leftarrow S^{\beta_1}(-q-1) \leftarrow S^{\beta_2}(-q-2) \leftarrow \cdots$$
$$\cdots \leftarrow S^{\beta_{e-2}}(-e-q+2) \leftarrow S^{\beta_{e-1}}(-e-q+1) \leftarrow S(-e-2q) \leftarrow 0$$

with $\beta_p := \binom{p+q-1}{q}\binom{e+q}{p+q} - \binom{e+q-p-1}{q-1}\binom{e+q-1}{e+q-p}$.

Thank you for your attention!