2021 Symposium in Algebraic Geometry
(in memory of prof. Bumsig Kim)

"Hypersurfaces with defect"

전주대학교 - 수학교육과

홍규식

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The Equivariant Schubert Cycles

by

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Approved by

Major Advisor

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홍 규 식

위 논문은 포항공과대학교 대학원 석사 학위논문으로 학위논문 심사위원회를 통과하였음을 인정합니다.

2002년 12월 30일

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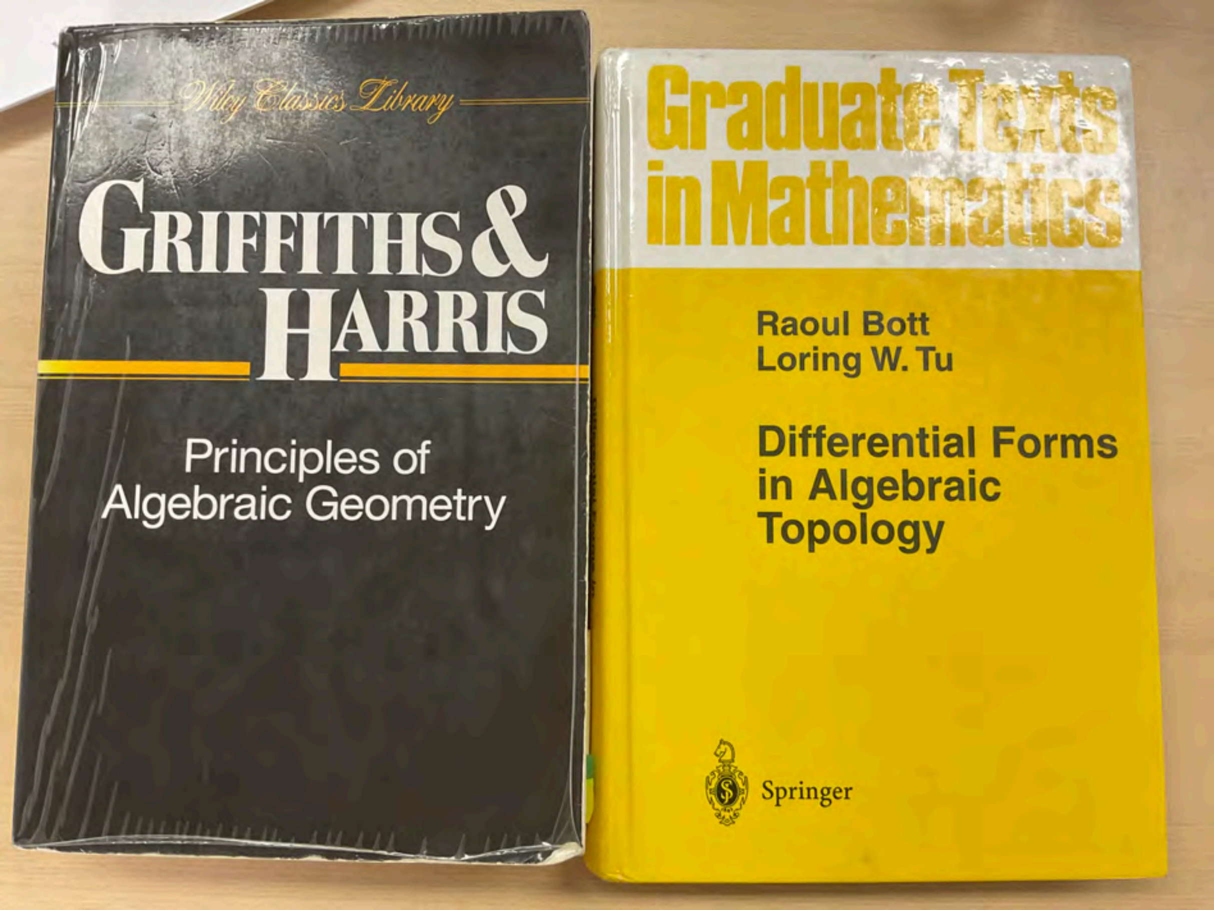
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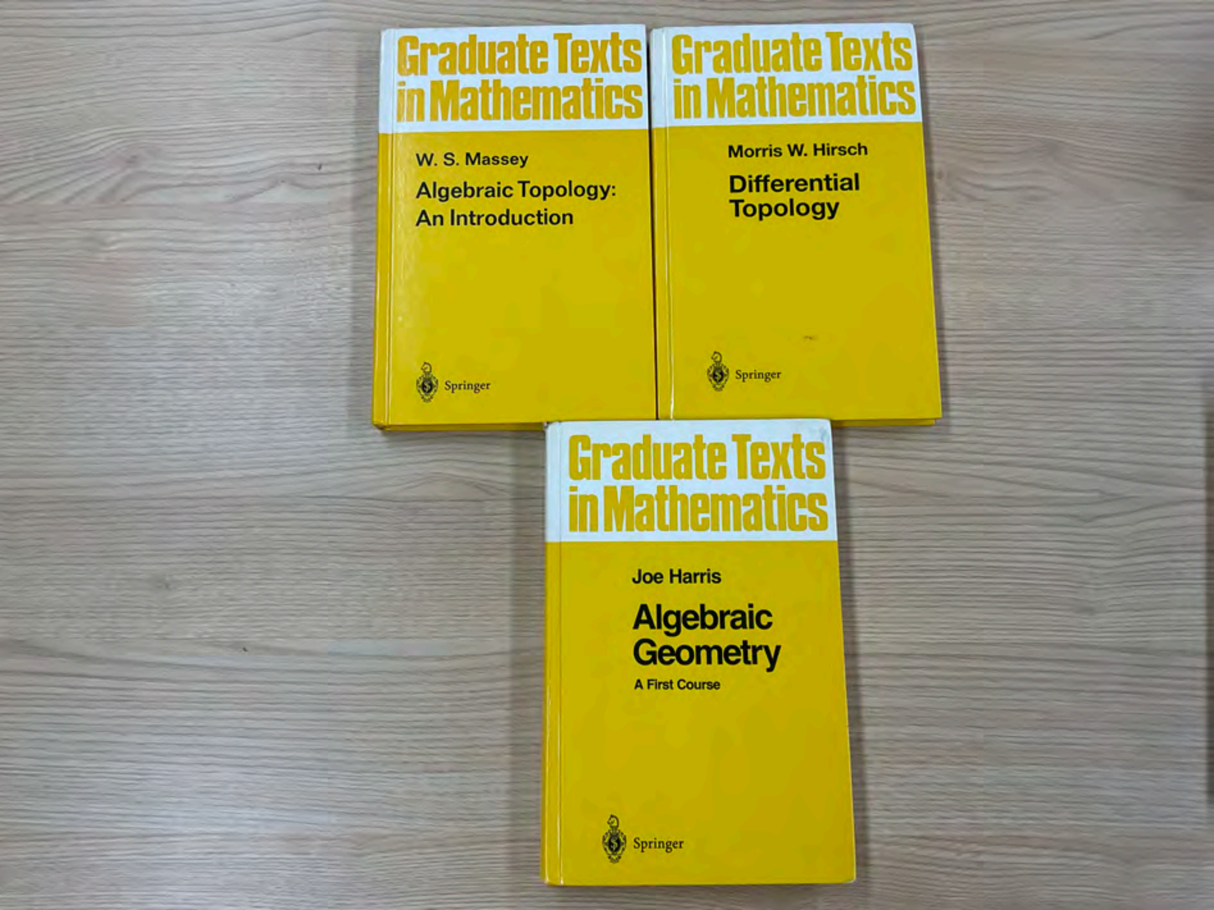
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X ; proj. hypersurface $\subset \mathbb{P}^n_k$

k : field, $n \geq 3$; integer

X has "defect"

if $h^i(X) \neq h^i(\mathbb{P}^n_k)$

for some $i \in \{n, \cdots, 2n-2\}$,

where h^n ; n-th Betti number

in a resonable cohomology theory

for k-varieties.

* resonable cohomology theories

- singular, algebraic de Rham or Kähler-de Rham cohomology
  (char $k = 0$)
- rigid cohomology (k : ~~perfect field~~ of char $k > 0$)
- étale cohomology

Note

X : hypersurface $\subset \mathbb{P}^n$

has defect $\Leftarrow$ ① singular ! (Example 1)

② have "many" singularities compared to their degree. (Example 2)

"Example 1

Ⓕ

$$X_4 \subset \mathbb{P}^4_{\mathbb{C}} :$$

nodal quartic hypersurface

$$\Longrightarrow \quad h^4(\mathbb{P}^4) = 1 \quad \left(\because \ \mathrm{Pic}\,\mathbb{P}^4 \simeq \mathbb{Z}[\mathcal{O}_Y(1)] \right)$$

But $h^4(X_4) \geq 1 \rightsquigarrow h^4(X^4, \mathbb{Z}) \geq h^4(\mathbb{P}^4, \mathbb{Z})$.

Note: the defect of X_4 $\overset{}{\longleftrightarrow}$ 「매우관련」 the rank of the divisor class group

⑤

$$\sigma(X_4) = \operatorname{rank} \mathrm{Cl}(X_4) - \operatorname{rank} \mathrm{Pic}\, X_4$$

(defect of X_4) $= h^4(X_4) - 1 \geq 0$

$$\Longrightarrow \quad h^4(X_4) = 1 + \sigma(X_4) \geq 1 = h^4(\mathbb{P}^4)$$

$\therefore \quad \sigma(X_4) = 0 \iff h^4(X_4) = 1$ ~ ⊛ 「기하적의미 뒷장설명」

$\iff \operatorname{rank} \mathrm{Cl}(X_4) = \operatorname{rank} \mathrm{Pic}\, X_4$

$\iff X_4$ is $\mathbb{Q}$-factorial

$\iff X_4$ is factorial

Note X_4 : smooth $\Rightarrow$ By Poincaré duality on X_4, $\sigma(X_4) = 0$.

$$X_d \subset \mathbb{P}^n$$

; hypersurface of degree d

$$\sigma(X_d) = 0 \iff h^4(X_d) = h^4(\mathbb{P}^n) = 1$$
$$\iff \dim H^2(X_d, \mathbb{Z}) = 1$$
$$\iff S \subset X_d$$
$$\Rightarrow S = X_d \cap F$$
where $F \subset \mathbb{P}^n$ hypersurface

Example 2) $X_4 \subset \mathbb{P}^4$

nodal quartic hypersurface.

$\# | \operatorname{sing} X_4 | \leq 8$

$\Rightarrow \quad \sigma(X_4) = 0$

i.e., $\operatorname{rank} Cl(X_4) = \operatorname{rank} \operatorname{Pic}(X_4)$
$= \operatorname{rank} \operatorname{Pic}(\mathbb{P}^4)$

② have "many" singularities compared to their degree. (Example 2)

Note; $\# | \operatorname{sing} X_d | < (d-1)^2 \Rightarrow \sigma(X_d) = 0$

By Clemens (1983) + Cynk (2001), ...

Hodge numbers of double solids

$$\operatorname{rank}(H_4(X_d,\mathbb{Z})) = \underbrace{1}_{h^4(\mathbb{P}^4)} + \delta_{X_d},$$

where $\delta_{X_d} = h^0(\mathcal{O}_{\mathbb{P}^4}(2d-5)\otimes \mathcal{I}_{\mathrm{Sing}(X_d)})$

$$- [\, h^0(\mathcal{O}_{\mathbb{P}^4}(2d-5)) - \#|\mathrm{Sing}(X_d)| \,]$$

, $X_d \subset \mathbb{P}^4$; nodal hypersurface.

(Rams, 2008)

「A-D-E singularities 갖는 초곡면의」 defect 「정리·계산」

in Complex proj. normal Cohen-Macaulay 4-fold

「Global Topological problem

Certain homogeneous forms and Vanishing on finite number of pts of $\mathbb{P}^n$.」

Part A, $\delta(X)=0$

"Zero defect"

$X \Rightarrow$

"3-folds with nodes,"

- * Double solids (ex. quantic double solids)
- * nodal quantic hypersurfaces $\subset \mathbb{P}^4$
- * nodal hypersurfaces $\subset \mathbb{P}^4$
- * nodal complete intersection 3-folds $\subset \mathbb{P}^5$ (NCIT)

"k-folds with nodes"

- * nodal hypersurfaces $\subset \mathbb{P}^n$

X : 3-fold, $\delta(X) = 0 \iff X$: factorial

* Double solids ;

X_{2r} double cover of $\mathbb{P}^3$ ramified along a surface $S \subset \mathbb{P}^3$ of degree $2r$.

(2006, Cheltsov, Park, —)

$$\#|\mathrm{Sing}\, X_4| \leq 5 \implies \sigma(X_4) = 0$$

$$\#|\mathrm{Sing}\, X_6| \leq 14 \implies \sigma(X_6) = 0$$

⊕ "Rationality of quartic double solids"

(Tihomirov, 1982, Voisin, 1988, Cheltsov, 2019 …)

sm X_4 is irrational.

Conj, X_4 : factorial $\Rightarrow X_4$: irrational

X; 3-fold, $\delta(X)=0 \Leftrightarrow X$: factorial

* nodal hypersurface $\subset \mathbb{P}^4$

V_d of degree d

$$\#|\mathrm{sing}\, V_d| < (d-1)^2 \Rightarrow \delta(V_d)=0$$

* "Rationality of quartic hypersurfaces $\subset \mathbb{P}^4$"
V_4

Note V_4 factorial $\Rightarrow V_4$: irrational (Manin, Mella, ...)

$$\#|\mathrm{sing}\, V_4| \lesssim 8 \Rightarrow \sigma(V_4)=0$$
$$\Rightarrow V_4 : \text{irrational}$$

X ; 3-fold, $\delta(X) = 0 \Leftrightarrow X$; factorial

(Conjecture)

$G_n, \underset{\text{smooth}}{G_m} \subset \mathbb{P}^5$; hypersurfaces,

$NCIT(n,m) = G_n \cap G_m$

; nodal complete intersection threefold

$$\#|\operatorname{Sing} NCIT(n,m)| < (n+m-2)^2 - (n-1)(m-1)$$

$$\Rightarrow \delta(NCIT(n,m)) = 0$$

" (2009, Kosta) * $|\operatorname{Sing} NCIT(n,m)| \leq (n+m-2)^2 - (n+m-2)(m-1) - 1 \Rightarrow \delta(\cdot) = 0$ "

(With ス人ス, ... Works in Progress. 2021.....)

「Rmk」 Defect 값은, 어떤 경우 매우 민감히 반응

<u>Example 3</u> $V_4 \subset \mathbb{P}^4$; nodal quartic hypersurface

① $\#|\mathrm{Sing}\, V_4| \leq 8 \Rightarrow \sigma(V_4) = 0$

② $9 \leq \#|\mathrm{Sing}\, V_4| \leq 35$

「$h_1 \cdot g_3 + f_2 \cdot k_2 = 0 \Rightarrow \sigma(V_4) \neq 0$」

민감 반응!

③ $\#|\mathrm{Sing}\, V_4| > 35 \Rightarrow \sigma(V_4) \geq 1$

$35 = \dim H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(3))$

Example F) $\underline{NCIT(n,m)}$; ① $G_n \cap G_m \subset \mathbb{P}^5$ ② G_m: Smooth ③ nodal

$$\#\left|\operatorname{Sing} NCIT(n,m)\right|$$

$$> \dim H^0(\mathcal{O}_{\mathbb{P}^5}(2n+m-6)|_{G_m})$$

$$\Rightarrow \sigma(\operatorname{Sing} NCIT(n,m)) \geq 1.$$

「민감 반응 Defect!」

$$* \quad (n+m-2)^2-(n-1)(m-1)$$

$$\leq \#\left|\operatorname{Sing} NCIT(n,m)\right|$$

$$\leq \dim H^0(\mathcal{O}_{\mathbb{P}^5}(2n+m-6)|_{G_m})$$

hypersurface $T \subset \mathbb{P}^n$, $\sigma(T) = 0$

$*$ k-fold, $(k \geq 4)$

Open!

Part B, $\sigma(X) \geq 1$

" positive defect "

$X \rightsquigarrow$

3-fold (* hypersurfaces $\subset \mathbb{P}^4$

k-fold (* global Tjurina number of X

$\tau(X)$

X ; 3-fold, $\sigma(X) \geq 1$

* nodal hypersurface $V_d \subset \mathbb{P}^4$ of degree d

$$\#|\operatorname{Sing} V_d| > \operatorname{rank} H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2d-5))$$

$$\Rightarrow \quad \sigma(V_d) \geq 1$$

X ; 3-fold, $\sigma(X) \geq 1$

Example 5

$V_4 \subset \mathbb{P}^4$,

very general (i.e., from the complement of a countable union of Zariski closed subsets in moduli)

quantic 3-fold containing a plane $\pi \subset \mathbb{P}^4$.

$\sigma(V_4)=1$

$$\Longrightarrow \quad x h_3(x,y,z,t,w) + y g_3(x,y,z,t,w) = 0 \subset \mathrm{Proj}(\mathbb{C}[x,y,z,t,w]).$$

「h_3, g_3 ; homog. poly. of degree 3, $\pi := \{x=y=0\}$, $\# |\mathrm{Sing}\, V_4| = 9$.

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$\Rightarrow$ $Cl(V_4) = \mathbb{Z} \oplus \mathbb{Z}$

$Cl(V_4) / Pic(V_4)$

$\because$ ① $\#|\mathrm{Sing}\, V_4| \leq 8 \Rightarrow \sigma(V_4) = 0$

② $\#|\mathrm{Sing}\, V_4| = 9$ + $\sigma(V_4) \neq 0$ $\pi C V_4$

$\Rightarrow$ $\underbrace{\sigma(V_4)}_{1 \text{ (from ②)}} = \mathrm{rank}\, Cl(V_4) - \underbrace{\mathrm{rank}\, Pic(V_4)}_{1}$

Example 6

$\ulcorner \sigma(V_4) = 1 = \underline{h^{4}(V_4) - 1}$, with $h^{4} \Leftarrow 2$ $\lrcorner$

$V_4 \subset \mathbb{P}^4$; sufficiently general quartic 3-fold containing a sm. del Pezzo surface $S \subset \mathbb{P}^4$ of degree 4.

$$\Longrightarrow \quad a_2(x,y,z,t,w)\,h_2(x,y,z,t,w) + b_2(x,y,z,t,w)\,g_2(x,y,z,t,w) = 0 \subset \mathrm{Proj}(\mathbb{C}[x,y,z,t,w])$$

Here a_2, h_2, b_2, g_2 : homog. poly. of degree 2,

$S := \{h_2 = g_2 = 0\}$, $\#|\mathrm{Sing}\, V_4| = 16$.

$\Longrightarrow \quad Cl(V_4) = \mathbb{Z} \oplus \mathbb{Z}$

$(\because$ $f : U \to \mathbb{P}^4$; blow-up of S,

E : exceptional divisor of the birational map f.

$H = f^*(\mathcal{O}_{\mathbb{P}^4}(1)))$

$\Longrightarrow \quad |2H-E|$: has no base pts.

($\because$ del Pezzo surface $S \subset \mathbb{P}^4$ is a complete intersection of two quadrics)

$\Longrightarrow$ In particular, $\begin{cases} 2H-E \text{ ; nef} \\ 4H-E \text{ ; ample.} \end{cases}$

$\tilde{V}_4 \subset U$; a proper transform of V_4

$\Rightarrow$ $\tilde{V}_4$ is rationally equi. to the divisor $4H-E$ on the 4-fold U.

$\Rightarrow$ $f|_{\tilde{V}_4} : \tilde{V}_4 \longrightarrow V_4$; small resolution,
$\tilde{V}_4$: smooth

$\Rightarrow$ By the Lefschetz thm,

$$H^2(\tilde{V}_4, \mathbb{Z}) \cong H^2(U, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

X ; 3-fold, bounds for $\sigma(X)$

Theorem (2011, Anne-Sophie Kaloghiros)

「$\sigma(V_4) \leq 15$」

$V_4 \subset \mathbb{P}^4$: quartic hypersurface with terminal singularities,

① rank $Cl(V_4) \leq 9$ when V_4 contains neither a plane nor a quadric,

② rank $Cl(V_4) \leq 10$ when V_4 does not contain a plane

③ rank $Cl(V_4) \leq 16$, with equality precisely when Y is projectively equi. to the Burkhardt quartic.

「Works in progress」

$V_4 \subset \mathbb{P}^4$; nodal quartic hypersurface

When $9 \leq \#|\mathrm{Sing}\, V_4| < 45$.

$*$ $\operatorname{rank} Cl(V_4) = ?$

「Note」 $\#|\mathrm{Sing}\, V_4| = 45 \Rightarrow \operatorname{rank} Cl(V_4) = 16$

$\Rightarrow \sigma(V_4) = 15$

Tjurina number of hypersurfaces with defect

Thm (2020, Linder)

K ; field, $\mathrm{Char}\, K = 0$.

Suppose that $X \subseteq \mathbb{P}^n_K$, $n \geq 3$, is a hypersurface with defect in algebraic de Rham, Kähler-de Rham, singular or étale cohomology.

$\Longrightarrow$ the global Tjurina number $\tau(X)$ of X

$$\tau(X) \geq \frac{\deg(X) - n + 1}{n^2 + n + 1}.$$

Q) hypersurface $\subset Gr(2,4)$ with スイス

감사합니다.