Jordan constants of quaternion algebras over number fields and simple abelian surfaces over fields of positive characteristic

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- Introduction / History (to my best knowledge)
- Jordan constants: quaternion algebras over number fields
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Key notions of this talk (intro. by Prof. V. Popov):

- A group G is called a *Jordan group* if there exists an integer $d > 0$, depending only on G, such that every finite subgroup H of G contains a normal abelian subgroup A whose index $[H : A]$ is at most d.
- The minimal such an integer d is called the *Jordan constant of G* and is denoted by J_G.
- It is usually (very) difficult to compute the exact value of the Jordan constant J_G of a group G. (seems not many known examples)

Key notions of this talk (intro. by Prof. V. Popov):

- A group G is called a *Jordan group* if there exists an integer $d > 0$, depending only on G, such that every finite subgroup H of G contains a normal abelian subgroup A whose index $[H : A]$ is at most d.
- The minimal such an integer d is called the *Jordan constant of G* and is denoted by J_G.
- It is usually (very) difficult to compute the exact value of the Jordan constant J_G of a group G. (seems not many known examples)

Thus: our main topic today is the following

Problem

Given a group G (coming from a "Geometric" or "Arithmetic" object), is G a Jordan group, and if so, what is the (exact) value of the Jordan constant J_G?

Some fundamental examples: in these examples, $k = \overline{k}$ of char. 0 (unless otherwise specified).

- (C. Jordan - 1878): $\mathrm{GL}_n(\mathbb{C})$ (whence, $\mathrm{GL}_n(k)$) is a Jordan group for any $n \geq 1$.
  Cor: Every affine algebraic group is also a Jordan group.
- (M.J. Collins - 2007): Obtains Jordan constants of $\mathrm{GL}_n(k)$ for any $n \geq 1$. (e.g. $J_{GL_2(k)} = 60, J_{GL_3(k)} = 360$)

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* One non-example: $PGL_n(\overline{\mathbb{F}_p})$ ($n \geq 2$) for a prime p is not a Jordan group. (For later use to consider the Cremona group.)

More examples from Algebraic Geometry side: first, we recall:

Definition

For a smooth irreducible rational projective variety X of dim$= n$ over k, the group $\mathrm{Bir}(X)$, the group of birational automorphisms of X, is known as $\mathrm{Cr}_n(k)$, the Cremona group of rank n over k.

Then:

More examples from Algebraic Geometry side: first, we recall:

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Then:

- $\mathrm{Cr}_n(k)$ is a Jordan group for any $n \geq 1$.
  ($n = 1$ case: $\mathrm{Cr}_1(k) \cong PGL_2(k)$, whence Jordan,
  $n = 2$ case $\rightarrow$ J.P. Serre and
  $n \geq 3$ case $\rightarrow$ Y. Prokhorov and C. Shramov with aid of a result of C. Birkar)
- OTOH, for Jordan constants: only "known" for $n = 1, 2, 3$.
  ($n = 1$ case: $J_{\mathrm{Cr}_1(k)} = J_{PGL_2(k)} = 60$,
  $n = 2$ case $\rightarrow$ E. Yasinsky, $J_{\mathrm{Cr}_2(k)} = 7200$ and
  $n = 3$ case $\rightarrow$ Y. Prokhorov and C. Shramov, an upper bound)

For varieties X of small dimension ≤ 3, we also have:

- (V. Popov - 2011): for an irreducible algebraic variety X over k with $\dim X \leq 2$, the group $G := \mathrm{Bir}(X)$ is a Jordan group if and only if X is not birationally isom. to $\mathbb{P}^1_k \times E$ where E is an elliptic curve over k.

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- (Y. Prokhorov and C. Shramov - 2018) $\dim X = 3$ case: if X is a threefold over k, then the group $\mathrm{Bir}(X)$ is not Jordan if and only if X is birational either to $E \times \mathbb{P}^2$ (where E/k is an elliptic curve) or else to $S \times \mathbb{P}^1$ (where S is one of the followings: an abelian surface, a bielliptic surface, a surface of Kodaira dimension 1 such that the Jacobian fibration of the pluricanonical fibration $\phi : S \to B$ is locally trivial in Zariski topology).

* Just one more possibly interesting result is:

Theorem (Yuri Prokhorov and Constantine Shramov, 2016)

For every positive integer n, there exists a constant $J = J(n)$ such that for any rationally connected variety X of dimension n over an arbitrary field k of char. 0 and for any finite subgroup Γ of $Bir(X)$, there exists a normal abelian subgroup A of Γ with $[\Gamma : A] \leq J$.

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* For the automorphism group case, we record:
  - (M. Sheng and D.Q. Zhang - 2018): the full automorphism group of any projective variety over k is a Jordan group.

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Still, we have the following:

- (Fei Hu - 2020): proved two Jordan-type properties of the full automorphism group of any projective variety over a field of positive char.

More precisely: the first one is

Theorem (Fei Hu, 2020)

Let X/k be a projective variety with $k = \overline{k}$ and $char(k) = p > 0$. Then there exists a constant J_X, depending only on X, such that every p'-subgroup Γ of $Aut(X)$ contains a normal abelian subgroup A with $[\Gamma : A] \leq J_X$.

Here, a finite group is called a *p'-group* if its order is relatively prime to p.

To introduce the second one, we first recall one result of Larsen and Pink, and then give one related notion:

Theorem (Michael Larsen and Richard Pink, 2011)

For any $n \in \mathbb{Z}_{>0}$, there exists a constant $J'(n)$ such that any finite subgroup Γ of GL_n over a field k of char. $p > 0$ contains a normal abelian p'-subgroup A with $[\Gamma : A] \leq J'(n) \cdot |\Gamma_{(p)}|^3$.

Kinda motivated by this result, we give

Definition

Let p be a prime. A group G is called a *p-Jordan group* if there exist two constants $J'(G)$ and $e(G)$, depending only on G, such that every finite subgroup Γ of G contains a normal abelian p'-subgroup A with $[\Gamma : A] \leq J'(G) \cdot |\Gamma_{(p)}|^{e(G)}$.

Here, $\Gamma_{(p)}$ denotes the Sylow p-subgroup of Γ.

Then the second of the aforementioned Jordan-type properties is given in the following

Theorem (Fei Hu, 2020)

Let X/k be a projective variety with $k = \overline{k}$ and $char(k) = p > 0$. Then there exists a constant J'_X and e_X, depending only on X, such that every finite subgroup Γ of $Aut(X)$ contains a normal abelian p'-subgroup A with $[\Gamma : A] \leq J'_X \cdot |\Gamma_{(p)}|^{e_X}$.

Here, $\Gamma_{(p)}$ denotes the Sylow p-subgroup of Γ.

* In other words, the group $\mathrm{Aut}(X)$ is a p-Jordan group.

Introduction - this talk

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According to history, we are naturally led to consider "what happens in terms of the "classical" Jordan property and Jordan constants in positive char. coming from "geometric" or "arithmetic" objects?"

The following is a specific case:

Question (Main Question=MQ, in this talk)

Let X be a simple abelian surface over a field k of positive char.

1. *Let $D = End_k^0(X) := End_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Is $D^\times$ a Jordan group? If so, what is $J_{D^\times}$?*
2. *Is $G := Aut_k(X)$ a Jordan group? If so, what is J_G?*

Along this direction, at least to me, there was nothing known.

In this talk, we try to (briefly) answer the MQ by using a corresponding result on the Jordan constants of quaternion (division) algebras D over number fields K with $[K : \mathbb{Q}] \leq 2$.

Introduction - this talk

"Main players in this talk:"

General Setting in this talk:

- K, a number field with $[K : \mathbb{Q}] \leq 2$
- D, a quaternion division algebra over K
- k, a field of positive characteristic (later specified in this talk)
- G, (an infinite) group
- X, a simple abelian surface over k (i.e. a complete algebraic variety over k of dimension 2 with $X(k)$ forming an abelian group, whose only abelian subvarieties are 0 and X itself)
- $\mathrm{End}^0_k(X) := \mathrm{End}_k(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, the endomorphism algebra of X over k
- $\mathrm{End}_k(X)$, the endomorphism ring of X over k
- $\mathrm{Aut}_k(X)$, the group of automorphisms of X over k

Introduction - this talk

We finish this introductory part by recalling one fact on the endomorphism algebra of X for the case when the base field k is finite, following "Albert's classification":

Lemma

If X is a simple abelian surface over a finite field k, then $D := End_k^0(X)$ is of one of the following three types:
(i) D is a totally definite quaternion algebra over either $\mathbb{Q}$ or a real quadratic field;
(ii) D is a CM-field of degree 4;
(iii) D is a quaternion division algebra over an imaginary quadratic field.

* This is why we consider the Jordan property / Jordan constant of quaternion division algebras over number fields of small degree first in this talk, as we will do from the next slide.

Jordan constants: quaternion algebras over number fields

* Now, to answer MQ-1), we begin with the following:

Lemma (Lem A, Elementary but useful lemma)

Let G be a Jordan group. Then every subgroup $H \leq G$ is a Jordan group and we have

$$J_G = \sup_{H \leq G} J_H$$

where the supremum is taken over all finite subgroups H of G.

Jordan constants: quaternion algebras over number fields

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* Application: If every finite subgroup of a Jordan group G is cyclic, then $J_G = 1$.

Example (related to the above appl.: Koras-Russell cubic 3fold)

Let k be an alg. closed field of char. zero, and let $X \subseteq \mathbb{A}_k^4$ be the nonsingular hypersurface defined by $x_1^2x_2 + x_3^2 + x_4^3 + x_1 = 0$. Then it can be shown that every finite subgroup of $\mathrm{Aut}(X)$ is cyclic, and hence, we have $J_{\mathrm{Aut}(X)} = 1$.

Jordan constants: quaternion algebras over number fields

Two main auxiliary results (that are interesting in their own right):

Theorem (H, 2021? - summary)

Let $\mathcal{D}$ be a quaternion division algebra over a number field K with $[K:\mathbb{Q}] \leq 2$. Then $\mathcal{D}^\times$ is Jordan, and we can "precisely" compute $J_{\mathcal{D}^\times}$ as follows:

1. *($K = \mathbb{Q}$ case) $J_{\mathcal{D}^\times} \in \{1, 2, 12\}$ depending on $Ram(\mathcal{D})$.*
2. *($[K:\mathbb{Q}] = 2$ case) $J_{\mathcal{D}^\times} \in \{1, 2, 12, 24, 60\}$ depending on K and $Ram(\mathcal{D})$ (for corresponding K).*

Here, $\mathrm{Ram}(\mathcal{D})$ is the set of all primes of K at which $\mathcal{D}$ is ramified.

* We recall: Let ν be a prime (place) of K. Then $\mathcal{D}$ is ramified at ν if $\mathcal{D} \otimes_K K_\nu$ is a quaternion division algebra over K_ν.

Jordan constants: quaternion algebras over number fields

In fact, there is a complete description for $J_{\mathcal{D}^\times}$ in these cases.

* ($K = \mathbb{Q}$ case)

Theorem (Thm A, H, 2021?)

Let $\mathcal{D}$ be a quaternion division algebra over $\mathbb{Q}$ and let $R_{\mathcal{D}} = Ram(\mathcal{D})$. Then the group $\mathcal{D}^\times$ is Jordan, and we have

$$J_{\mathcal{D}^\times} = \begin{cases} 12 & \text{if } R_{\mathcal{D}} = \{2, \infty\}; \\ 2 & \text{if } R_{\mathcal{D}} = \{3, \infty\}; \\ 1 & \text{otherwise.} \end{cases}$$

* Finite groups that matter in this case: $\mathfrak{T}^*$ and Dic_{12}

* (Slightly harder $[K : \mathbb{Q}] = 2$ case)

Theorem (Thm B, H, 2021?)

Let $\mathcal{D}$ be a quat. division alg. over a quad. number field K, $R_{\mathcal{D}} = Ram(\mathcal{D})$, and let R_∞ be the set of all inf. places of K. Also, let $d > 0$ be a square-free int. Then $\mathcal{D}^\times$ is Jordan, and we have

$$J_{\mathcal{D}^\times} = \begin{cases} 60 & \text{if } K = \mathbb{Q}(\sqrt{5}) \text{ and } R_{\mathcal{D}} = R_\infty; \\ 24 & \text{if } K = \mathbb{Q}(\sqrt{2}) \text{ and } R_{\mathcal{D}} = R_\infty; \\ 12 & \text{if } \mathcal{D} = D_{2,\infty} \otimes_{\mathbb{Q}} K,\ K = \mathbb{Q}(\sqrt{d}) \text{ with } d \neq 2, 5, \text{ or} \\ & \mathcal{D} = D_{2,\infty} \otimes_{\mathbb{Q}} K,\ K = \mathbb{Q}(\sqrt{-d}) \text{ with } d \equiv 7 \pmod 8; \\ 2 & \text{if } \mathcal{D} = D_{3,\infty} \otimes_{\mathbb{Q}} K,\ K = \mathbb{Q}(\sqrt{d}) \text{ with } d \equiv 9, 17 \pmod{24} \\ & \mathcal{D} = D_{3,\infty} \otimes_{\mathbb{Q}} K,\ K = \mathbb{Q}(\sqrt{-d}) \text{ with } d \equiv 2 \pmod 3; \\ 1 & \text{otherwise.} \end{cases}$$

* Finite groups that matter in this case: $\mathfrak{I}^*, \mathfrak{O}^*, \mathfrak{T}^*$, dicyclic groups, and cyclic groups

Jordan constants: quaternion algebras over number fields

As a special case of Thm B, we record the following result for our later use:

Corollary (H, 2021?)

Assume that $\mathcal{D} = D_{p,\infty} \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{p})$ for some prime p. Then:

$$J_{\mathcal{D}^\times} = \begin{cases} 60 & \text{if } p = 5; \\ 24 & \text{if } p = 2; \\ 12 & \text{if } p \equiv 3 \pmod 4 \text{ or } p > 5 \text{ and } p \equiv 5 \pmod 8; \\ 2 & \text{if } p \equiv 17 \pmod{24}; \\ 1 & \text{otherwise.} \end{cases}$$

* Very naive idea of proofs: Explicit list of "maximal" finite subgroups of various quat. div. algs. over corresponding number fields + Theory of Brauer groups + Use of Lem A.

Application: Jordan constants of simple abelian surfaces

Here comes an application of our previous results to the theory of abelian varieties in positive char., that gives an answer for MQ-1).

- (case: base field k being finite)

Let $q = p^a$ for some prime p and an integer $a \geq 1$ in the following theorem. Then:

Theorem (Thm C, H, 2021?)

1. *Let X be a simple abelian surface over a finite field $k = \mathbb{F}_q$, and let $D = \mathrm{End}_k^0(X)$. Then the Jordan constant $J_{D^\times}$ of $D^\times$ is contained in the set $\{1, 2, 12, 24, 60\}$. Conversely,*
2. *Let n be an integer contained in the set $\{1, 2, 12, 24, 60\}$. Then there is a simple abelian surface X over some finite field $k = \mathbb{F}_q$ such that $J_{D^\times} = n$, where $D = \mathrm{End}_k^0(X)$.*

Application: Jordan constants of simple abelian surfaces

Proof.

Idea of proof: (For part (1) above:)

- We know that D is either i) a tot. definite quat. alg. over $\mathbb{Q}$ or a real quad. field, or ii) a quat. div. alg. over an imag. quad. field, or iii) a CM-field of degree 4.
- Use our previous two auxiliary theorems for i) and ii).

(For part (2) above:)

- Choose an integer $n \in \{1, 2, 12, 24, 60\}$, and in turn, choose a suitable prime p (corresponding to the chosen n).
- Use Honda-Tate theory and ramification theory of number fields to construct a simple abelian surface over $k = \mathbb{F}_q$ ($q = p^a$) with $D = D_{p,\infty} \otimes_{\mathbb{Q}} \mathbb{Q}(\sqrt{p})$.
- Use the previous Corollary.

□

Application: Jordan constants of simple abelian surfaces

Another way to view Thm C is the following

Remark

Let S be a set of pairs (n, p) of an integer $n \geq 1$ and a prime p with the property that there exists a simple abelian surface X over a finite field $k = \mathbb{F}_q$ with $q = p^a$ $(a \geq 1)$ and $J_{D^\times} = n$ where $D = \mathrm{End}_k^0(X)$. If we choose an element (n, p) randomly from S, then it is most probable that $n = 12$ in the sense of the prime number theorem for arithmetic progression.

Idea of proof.

We can see that $(1, p) \in S$ iff $p \equiv 1 \pmod{24}$, $(2, p) \in S$ iff $p \equiv 17 \pmod{24}$, $(12, p) \in S$ iff $p = 3$ or $p \equiv 7, 11, 13, 19, 23 \pmod{24}$ or else $p \equiv 5 \pmod{24}$ with $p > 5$, $(24, p) \in S$ iff $p = 2$, and $(60, p) \in S$ iff $p = 5$. □

Application: Jordan constants of simple abelian surfaces

- (case: base field k being alg. closed of positive char.)

As a preliminary, we recall:

Proposition

Let X be a simple abelian surface over an alg. closed field k of characteristic $p > 0$. Then the endomorphism algebra $\mathrm{End}_k^0(X)$ is of one of the following types:
(1) $\mathbb{Q}$; (2) a real quadratic field; (3) a CM-field of degree 4; (4) an indefinite quaternion algebra over $\mathbb{Q}$.

Then we can obtain (via a similar but slightly different argument):

Theorem (Thm D, H, 2021? - seems interesting to me)

Let n be an integer and let $p > 0$ be a prime. Then there exists a simple abelian surface X over some alg. closed field k of char. p such that $J_{D^\times} = n$, where $D = \mathrm{End}_k^0(X)$ if and only if $n = 1$.

Application: Jordan constants of simple abelian surfaces

Using the fact that $\mathrm{Aut}_k(X)$ is a subgroup of the multiplicative subgroup of $\mathrm{End}_k^0(X)$, we have the following result on the exact value of the "classical" Jordan constant:

Corollary

Let X be a simple abelian surface over an alg. closed field k of characteristic $p > 0$. Then we have $J_{Aut_k(X)} = 1$.

Application: Jordan constants of simple abelian surfaces

Here comes a similar result for special higher dimensional cases: let $g \geq 3$ be a prime, and let X be a simple abelian variety of dimension g over an alg. closed field k with $\mathrm{char}(k) = p > 0$. Further, let $D = \mathrm{End}_k^0(X)$. Then we have:

Remark

1. *D is of one of the following types:*
   (i) D is a field : $D = \mathbb{Q}$ or D is a tot. real field of deg. g or D is an imag. quad. field, or else D is a CM-field of deg. $2g$;
   (ii) $D = D_{p,\infty}$ if (and only if) $g \geq 5$;
   (iii) D is a central simple division algebra of degree g over an imag. quad. field (and the p-rank of X equals 0).
2. *In view of item (1), we can see that $J_{D^\times} = 1$ if $g = 3$, and $J_{D^\times} \in \{1, 2, 12\}$ if $g \geq 5$.*

Application: Jordan constants of simple abelian surfaces

We end this talk by giving some questions (, some of which might be interesting only to me):

Question

1. *Compute the exact value of* $J_{Cr_n(k)}$ *for* $n \geq 3$ *and* k *with* $char(k) = 0$.
2. *Is* $Cr_n(k)$ *a* p*-Jordan group for* $n \geq 2$ *and a field* k *of char.* $p > 0$*? [Fei Hu]*
3. *For* X/k *a simple abelian variety of dimension* $g \geq 3$ *(e.g.* $g = 4$*) with* $char(k) = p > 0$*, what is the exact value of* $J_{Aut_k(X)}$*?*
4. *How about other projective varieties in either char. 0 or positive char. (in terms of the Jordan constants) ?*

etc.

Thank you for your attention!

Delivered from a Naver blog