SHARP CONVERGENCE RATE OF CURVE SHORTENING FLOW

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Abstract. The classical result of Gage-Hamilton shows every curve shortening flow of convex closed curve converges to a round circle after rescaling. The goal of this three-hour lecture series to answer how fast the solution converges to the circle and introduce its application to the regularity of arrival time.

More precisely, Gage-Hamilton says if $\Gamma_t \subset \mathbb{R}^2$, for $t \in (-T, 0)$, is a curve shortening flow that shrinks to the origin at time $t = 0$, then the rescaled flow

$$\tilde{\Gamma}_\tau = (-t)^{-1/2}\,\Gamma_t \quad \text{for} \quad \tau = -\log(-t)$$

converges to $\sqrt{2}S^1$, the circle of radius $\sqrt{2}$, as $\tau \to \infty$. We will show $\tilde{\Gamma}_\tau$ converges to $\sqrt{2}S^1$ with an exponential rate $e^{\delta\tau}$ for $\delta < 0$. Moreover, the optimal δ and asymptotic profile of $\tilde{\Gamma}_\tau$ are dictated by the linear analysis and spectral theory around the stationary circle. This idea based on linear technique is key to recent developments in the analysis of singularities in nonlinear geometric equations. We present this technique in a simple context of curve shortening flow.

1. Preliminaries and set-up of problem

We say 1-parameter family of smooth closed embedded curves $\{\Gamma_t\}_{t\in[0,T)}$ in $\mathbb{R}^2$ is a solution to curve shortening flow (CSF) if

$$\frac{\partial}{\partial t}F(p,t) = \frac{\partial^2}{\partial s^2}F(p,t).$$

Here s is an arc-length parameter. Using outer unit normal ν and the curvature κ with respect to $-\nu$, one can write $\partial^2_{ss}F = -\kappa\nu$. This note we only consider the flow of embedded closed curves.

The classical theorem of Gage and Hamilton shows

Theorem 1.1 (Convex curve becomes round [1]). *If Γ_0 is convex, then smooth unique solution Γ_t exists until Γ_t shrinks to a point in finite time $T < \infty$. Moreover, the convexity is preserved (Γ_t is convex) and Γ_t smoothly converges to a round circle if we rescale around the shrinking point.*

To understand the convergence statement precisely, let $A(t)$ be the enclosed area by Γ_t. By Gauss-Bonet, $\frac{d}{dt}A = -2\pi$. Therefore, provided that the solution converges to a point (as stated in the theorem), the extinction time is $T = \frac{A(0)}{2\pi}$.

After translating time and space, suppose Γ_t is defined on $[-T, 0)$ and shrinks to the origin. Then $A(t) = -2\pi t$. If we rescale Γ_t by $1/\sqrt{-t}$ as $\tilde{\Gamma}_t = \frac{1}{\sqrt{-t}}\Gamma_t$. Then the enclosed volume will be constant 2π. Therefore [1] proves $\tilde{\Gamma}_t \to \sqrt{2}S^1$ smoothly as $t \to 0^-$. In principle, this should mean the smooth convergence of embeddings $(-t)^{-\frac{1}{2}}F(p,t)$ to an embedding of S^1 to a round circle, but this is equivalent to show that $\tilde{\Gamma}_t$ converges to S^1 as a graph. $\tilde{\Gamma}_t = \{r(\theta,t)(\cos\theta,\sin\theta) : \theta \in [0,2\pi]\}$ and $r(\cdot,t)$ converges to $\sqrt{2}$ in $C^\infty(S^1)$ as $t \to 0^-$.

Let us introduce a new time variable

$$\tau := -\log(-t), \quad \tau \in (-\log T, \infty).$$

Now, we can restate the theorem of Gage-Hamilton in our preferred setting.

Theorem 1.2 ([1] revisited)**.** *As* $\tau \to \infty$,

$$e^{\frac{\tau}{2}} S(\theta,\tau) \longrightarrow \sqrt{2} \quad \text{in } C^\infty(S^1).$$

We are interested in the asymptotic behavior of

$$u(\theta,\tau) := e^{\frac{\tau}{2}} S(\theta,t) - \sqrt{2}.$$

Lemma 1.3 (Equation of $u(\theta,\tau)$)**.**

$$u_\tau = \frac{1}{2}u'' + u - \frac{(u+u'')^2}{2(\sqrt{2}+u+u'')} \tag{1.1}$$

Proof. Note $S_t = -(S+S_{\theta\theta})^{-1}$.

$$\begin{aligned} u_\tau &= e^{\frac{\tau}{2}}(S_t \tfrac{dt}{d\tau} + \tfrac{1}{2}S) = -\frac{1}{e^{\frac{\tau}{2}}(S+S_{\theta\theta})} + \frac{1}{2}e^{\frac{\tau}{2}}S \\ &= -\frac{1}{\sqrt{2}+u+u_{\theta\theta}} + \frac{1}{2}u + \frac{1}{\sqrt{2}} \\ &= \frac{u+u''}{\sqrt{2}(\sqrt{2}+u+u'')} + \frac{1}{2}u. \end{aligned} \tag{1.2}$$

The result follows from

$$\frac{1}{2} - \frac{1}{\sqrt{2}(\sqrt{2}+u+u'')} = \frac{u+u''}{2(\sqrt{2}+u+u'')}$$

□

We may decompose the equation into

$$u_\tau = Lu + Nu$$

where

$$Lu = u + \frac{1}{2}u'' \quad \text{and} \quad Nu = -\frac{(u+u'')^2}{2(\sqrt{2}+u+u'')}.$$

We say Lu is a linearization of (1.2) around 0. Nu represents the nonlinear error term and it should be noted that Nu is at least quadratic in u and its derivatives. Here is a formal (but very important) idea behind. $u = 0$ is a stationary solution to nonlinear equation (1.2). If another solution is very close 0, their difference would solve an approximate linear equation (which is like Taylor's first order approximation). Another direct way to see Lu is to compute formal derivative(variation) of among solutions. Suppose there is a 1-parameter family of solutions $u_s(\theta,\tau)$ with $s \in (-\epsilon,\epsilon)$ and $u_0 \equiv 0$. If we denote $\delta = \frac{\partial u}{\partial s}\big|_{s=0}$, then by taking the derivative of (1.2) w.r.t. s, we obtain

$$\delta_\tau = +\frac{\delta+\delta''}{\sqrt{2}^2} + \frac{1}{2}\delta = \delta + \frac{1}{2}\delta''.$$

As in Taylor's theorem, the remainder term Nu will become arbitrarily small when it is compared to Lu as u converges to 0. So the behavior of solution is expected to follow from the linear equation $u_\tau = Lu$.

In the linear case, the behavior directly follows from the spectral decomposition.

Example 1.4 (Linear heat equation). Let u be a solution to the heat equation $u_t = u''$ on $S^1 \times [0, \infty)$. Then $u(\cdot, 0) \in L^2(S^1)$ can be decomposed by Fourier basis

$$u(\theta, 0) = c_0 + \sum_{k=1}^{\infty} c_k \cos(k\theta) + s_k \sin(k\theta)$$

where

$$c_k = \frac{\langle u(0), \cos(k\theta)\rangle}{\|\cos(k\theta)\|_{L^2}^2} = \frac{\int u(\theta, 0) \cos k\theta d\theta}{\pi}$$

and

$$c_0 = \frac{\langle u(0), 1\rangle}{\|1\|_{L^2}^2} = \frac{\int u(\theta, 0) \cos k\theta d\theta}{2\pi}.$$

Note $\cos(k\theta)$ (including $1 = \cos 0\theta$) and $\sin(k\theta)$ are eigenfunction of $\Delta ='' $ with eigenvalue $-k^2$. Thus if $w = \varphi(t) \cos kt$ is a solution to the HE, then

$$\varphi' = -k^2\varphi \quad \Longrightarrow \quad \varphi(t) = e^{-k^2 t}\varphi(0).$$

By the linearity, the solution has to be

$$u(t) = c_0 + \sum_k [c_k \cos(k\theta) + s_k \sin(k\theta)] e^{-k^2 t}.$$

One can expect the following results: (which is actually true)

$$\|u(t) - c_0\|_{L^2} = O(e^{-t}),$$
$$\|u(t) - c_0 - (c_1 \cos\theta + s_1 \sin\theta)e^{-t}\|_{L^2} = O(e^{-4t})$$

and similarly other higher order asymptotics. Here, in view of regularity theory for linear heat equation, one can even improve L^2 convergence to any $C^{k,\alpha}$ or $W^{k,p}$ convergence with the same convergence rates.

In our problem, it is straightforward to check the following:

Lemma 1.5 (Spectral decomposition)**.** *L is self-adjoint (unbounded) operator on $L^2(S^1)$ and $\{1, \cos(k\theta), \sin(k\theta)\}$ provides orthogonal basis of L^2 consisting of eigenfunctions $L\cos(k\theta) = (1 - k^2/2)\cos(k\theta)$.*

The first few eigenvalues are 1, 1/2, −1, −7/2 and so on. Without Nu, those eigenfunctions of positive eigenvalues, namely 1, $\cos\theta$, $\sin\theta$ tend to grow as $t \to \infty$. However, this case will be excluded by Theorem 1.2. We can formulate our main theorem.

Theorem 1.6. *Let u be a solution to* (1.1) *with $u \to 0$ in C^∞ as $\tau \to \infty$. Then either*
(1) there exist integer $k \geq 2$, and constants c_k, $s_k \in \mathbb{R}$ which are not both zero such that

$$\|u - (c_k \cos k\theta + s_k \sin k\theta)e^{(1-k^2/2)\tau}\|_{C^{m,\alpha}} = O(e^{(1-k^2/2-\epsilon)\tau}) \tag{1.3}$$

some $\epsilon > 0$ independent of k, m, α, or
(2) u converges to 0 faster than any exponential rates:

$$\|u\|_{C^{k,\alpha}} = O(e^{\lambda\tau}) \text{ for any } \lambda < 0.$$

Remark 1.7. It is interesting to know for given k, c_k, and s_k, if there exists a CSF which satisfies the asymptotic behavior (1.3). This is true. The construction of solutions with prescribed asymptotics is based on a fixed point theorem with iterations. See [2, Section 3].

In fact, the later case (2) happens only if Γ_t is round shrinking circle. i.e. $u \equiv 0$ for all t and θ. This is so called the unique continuation result and should be treated with separate interest. See [3, Section 4].

Theorem 1.8 (c.f. [3] Section 4)**.** *Suppose the second alternative in Theorem 1.6 holds. i.e.* $u(\tau)$ *decays faster than any exponential rates. Then* $u \equiv 0$*. i.e.* $\Gamma_t = \sqrt{-2t}S^1$.

2. MAIN RESULT

From now on at each τ*, we decompose* $u(t)$ *with basis and write with coefficients* $c_k(t)$ *and* $s_k(t)$

$$u(\theta,\tau) = \sum_k c_k(\tau)\cos k\theta + s_k(\tau)\sin k\theta.$$

Here k *goes from* 0 *and* $s_0(t) \equiv 0$.

We denote the projection of u *onto positive eigenspace by*

$$P^+u(\tau) = c_0(\tau) + c_1(\tau)\cos\theta + s_1(\tau)\cos\theta$$

and the projection onto negative eigenspace by

$$P^-u(\tau) = \sum_{k\geq 2} c_k(\tau)\cos k\theta + s_k(\tau)\sin k\theta = u - P^+u.$$

To treat Nu *as small quantity, we need*

Lemma 2.1. *There is uniform* C *such that, for large* $\tau > \tau_0$,

$$\begin{aligned} |\langle Nu(\tau), u(\tau)\rangle_{L^2}| &\leq C\|u(\tau)\|_{C^4}\|u(\tau)\|^2_{L^2} \\ |\langle Nu(\tau), P^+u(\tau)\rangle_{L^2}| &\leq C\|u(\tau)\|_{C^4}\|u(\tau)\|^2_{L^2} \\ |\langle Nu(\tau), P^-u(\tau)\rangle_{L^2}| &\leq C\|u(\tau)\|_{C^4}\|u(\tau)\|^2_{L^2} \end{aligned} \tag{2.1}$$

Proof. Let us write $Nu = -\frac{(u+u'')^2}{2(\sqrt{2}+u+u'')} = (u+u'')^2\rho$. May assume

$$\left\|\rho + \frac{1}{2\sqrt{2}}\right\|_{C^4} \leq \frac{1}{100}$$

by assuming $\tau > \tau_0$ sufficiently large.

$$\int (u+u'')^2\rho u \, d\theta = \int u^3\rho + 2u^2u''\rho + (u'')^2u\rho \, d\theta \tag{2.2}$$

Since the first two terms are bounded by $C\|u\|_{L^2}^2\|u\|_{C^4}$, it suffices to work with the last one. By the integration by parts,

$$\begin{aligned}\int (u'')^2 u\rho &= -\int u'(u''')u\rho + u'^2u''\rho + u'u''u\rho' \\ &= -\int \frac{1}{2}(u^2)'u'''\rho + \frac{1}{3}(u'^3)'\rho + \frac{1}{2}(u^2)'u''\rho' \\ &= \int \frac{1}{2}u^2(u'''\rho)' + \frac{1}{3}(u')^3\rho' + \frac{1}{2}u^2(u''\rho')^2 \\ &\leq C\|u\|_{L^2}^2\|u\|_{C^4} + \int \frac{1}{3}u'^3\rho'.\end{aligned} \tag{2.3}$$

Finally,

$$\begin{aligned}\int u'^3\rho' = -\int 2uu'u''\rho' + uu'^2\rho'' &= -\int (u^2)'(u''\rho' + \frac{1}{2}u'\rho'') \\ = \int u^2(u''\rho' + \frac{1}{2}u'\rho'') &\leq C\|u\|_{L^2}^2\|u\|_{C^4}.\end{aligned} \tag{2.4}$$

The second estimate is easier. This is because

$$\|\partial_\theta^\ell(P^+u)\|_{L^2} \leq C\|P^+u\|_{L^2}$$

for some uniform C depends only on ℓ. (Q: Why? A: it is finite dimensional.) We can actually choose $C = 1$. We leave it as an exercise. The third estimate follows from the first and second since $\langle Nu, P^-u\rangle = \langle Nu, u - P^+u\rangle$.

□

We would like to understand the behavior of solution $u(\tau)$ as a dynamics between coefficients c_k and s_k. Let

$$\begin{aligned}\partial_\tau\|P^+u\|_{L^2}^2 &= 2\langle P^+u, \partial_\tau P^+u\rangle_{L^2} = 2\langle P^+u, P^+\partial_\tau u\rangle_{L^2} \\ &= 2\langle P^+u, Lu + Nu\rangle_{L^2}\end{aligned} \tag{2.5}$$

Note

$$\langle P^+u, Lu\rangle = \langle P^+u, LP^+u\rangle \geq \frac{1}{2}\langle P^+u, P^+u\rangle$$

since $\lambda \geq 1/2$.

By a similar argument to P^-u, we obtain

$$\begin{aligned}\partial_\tau\|P^+u\|^2 &\geq +\|P^+u\|^2 - 2|\langle P^+u, Nu\rangle| \\ \partial_\tau\|P^-u\|^2 &\leq -2\|P^-u\|^2 + 2|\langle P^-u, Nu\rangle|.\end{aligned} \tag{2.6}$$

Lemma 2.2 (c.f. Merle-Zaag)**.** *Let $x(\tau)$, $y(\tau)$ be nonnegative absolutely continuous functions and $\lambda > 0$ is a constant that satisfy, for given $\epsilon > 0$, there is τ_0 such that for $\tau \geq \tau_0$*

$$\begin{aligned}x' &\geq +\lambda x - \epsilon y \\ y' &\leq -\lambda y + \epsilon x.\end{aligned} \tag{2.7}$$

If $x + y \to 0$ as $\tau \to \infty$, then there hold $x = o(y)$ as $\tau \to \infty$.

Proof. For $0<\epsilon<1$, suppose (2.7) holds for $\tau\geq\tau_0$. If we further assume $\epsilon<\lambda$,

$$\begin{aligned}(x-\epsilon y)' &\geq (\lambda x-\epsilon y)-\epsilon(-\lambda y+\epsilon x)=(\lambda-\epsilon^2)x+\epsilon(\lambda-\epsilon)y\\ &\geq(\lambda-\epsilon^2)(x-\epsilon y).\end{aligned} \tag{2.8}$$

This implies if $x-\epsilon y>0$ for some $\tau>\tau_0$, then $x-\epsilon y$ grows to ∞, which is a contradiction. Therefore, $x\leq\epsilon y$ for $\tau\geq\tau_1$. This proves the assertion. □

Lemma 2.1 and (2.6) *imply*

$$\begin{aligned}\partial_\tau\|P^+u\|^2 &\geq +\|P^+u\|^2-C(\|P^+u\|^2+\|P^-u\|^2)\|u\|_{C^4}\\ \partial_\tau\|P^-u\|^2 &\leq -2\|P^-u\|^2+C(\|P^+u\|^2+\|P^-u\|^2)\|u\|_{C^4}.\end{aligned}.$$

Since $\|u\|_{C^4}\to0$ *as* $\tau\to\infty$, *Lemma 2.2a pplies and we conclude*

$$\|P^+u\|_{L^2}=o(\|P^-u\|_{L^2}).$$

From the ODE of $\|P^-u\|_{L^2}$, *we get for all small* $\epsilon>0$, *there is* $\tau>\tau_0$ *large so that*

$$\partial_\tau\|P^-u\|^2_{L^2}\leq-2(1-\epsilon)\|P^-u\|^2_{L^2}$$

and this proves

$$\|P^-u\|_{L^2}=O(e^{(-1+\epsilon)\tau}).$$

Lemma 2.3 (Regularity improvement)**.** *Assume that* $\|u\|_{C^{k,\alpha}(S^1\times[\tau-2,\tau])}\leq/10000$, *then there is* $C=C(k,\alpha)$ *such that*

$$\|u\|_{C^{k,\alpha}(S^1\times[\tau-1,\tau])}\leq C\|u\|_{L^2(S^1\times[\tau-2,\tau])}.$$

Proof. By (1.2), $u_\tau=\frac{u+u''}{\sqrt{2}(\sqrt{2}+u+u'')}+\frac12u$ and we may view this equation as

$$u_\tau=au''+(a+\tfrac12)u$$

where $\|a-1/2\|_{C^{k-2,\alpha}}<1/10$. The result follows by the linear parabolic regularity theory. □

By the regularity improvement, we have the following:

Theorem 2.4 (First asymptotics)**.** $\|u(\tau)\|_{C^{k,\alpha}}=O(e^{(-1+\epsilon)\tau)})$ *as* $\tau\to\infty$ *for all* $\epsilon>0$.

Lemma 2.5. *Let* $x(t)$ *and* $f(t)$ *nonnegative functions satisfy* $x'\geq\lambda x-f$ *and* $|f|=O(e^{\lambda'(t)})$ *for* $\lambda'<\lambda$. *If* $x(t)=o(e^{\lambda t})$, *then* $x(t)=o(e^{\lambda' t})$.

Proof. We integrate this differential inequality and obtain, for $t_2\geq t_1$,

$$e^{-\lambda t_2}x(t_2)\geq e^{-\lambda t_1}x(t_1)-\int_{t_1}^{t_2}e^{-\lambda s}f(s)ds \tag{2.9}$$

and this yields

$$x(t_2)\geq e^{\lambda(t_2-t_1)}\left(x(t_1)-e^{\lambda t_1}\int_{t_1}^{\infty}e^{-\lambda s}f(s)ds\right).$$

This inequality implies that if $x(t_1) > e^{\lambda t_1}\int_{t_1}^{\infty} e^{-\lambda s} f(s)ds$, for some t_1, then $x(t_2)$ grows with a rate grater than $e^{\lambda t_2}$. Since this contradicts the assumption, we conclude

$$x(t) \le e^{\lambda t}\int_t^{\infty} e^{-\lambda s} f(s)ds = O(e^{\lambda' t}).$$

□

Lemma 2.6. *Let $y(t)$ and $f(t)$ nonnegative functions satisfy $y' \le \lambda y + f$ and $|f| = O(e^{\lambda'(t)})$ for $\lambda' < \lambda$. Then $y(t) = O(e^{\lambda t})$. If $y = \lambda y + f$, then there is $c \in \mathbb{R}$ such that*

$$y = ce^{\lambda t} + O(e^{\lambda' t}).$$

Proof. Similar to the previous lemma (in fact it is easier). □

Let us consider the evolution of $c_2(\tau) = \langle u(\tau), \cos 2\theta\rangle/\pi$.

$$(c_2)_\tau = \langle u_\tau, \cos 2\theta\rangle/\pi = (1 - 2^2/2)c_2 + \langle Nu, \cos 2\theta\rangle/\pi.$$

In view of Lemma 2.3, we now have

$$|\langle Nu, \cos 2\theta\rangle| \le C\|u\|_{C^2}^2 = O(e^{-2(1-\epsilon)\tau}).$$

By Lemma 2.6,

$$c_2(\tau) = c + O(e^{-2(1-\epsilon)\tau}) \quad s_2 = s + O(e^{-2(1-\epsilon)\tau})$$

for some c and s.

Proposition 2.7. *Suppose $\|u\|_{L^2} = O(e^{\lambda\tau})$ for some $\lambda < 0$ with*

$$1 - \frac{(j-1)^2}{2} > \lambda > 1 - \frac{j^2}{2}.$$

Then $\|u\|_{C^{k,\alpha}} = O(e^{(1-\frac{j^2}{2})\tau})$ and there are constants c and s such that

$$\|u - (c\cos j\theta + s\sin j\theta)e^{(1-\frac{j^2}{2})\tau}\|_{C^{k,\alpha}} = O(e^{(1-\frac{j^2}{2}-\epsilon)\tau}).$$

Proof. Let us $P_j^+ u$ be the projection of u onto the eigenspace of eigenvalues strictly less than $(1 - j^2/2)$ and $P_j^- u = u - P_j^+ u$.

Then by a similar argument as before,

$$\begin{aligned}\partial_\tau \|P_j^+ u\| &\ge (1 - \frac{(j-1)^2}{2})\|P_j^+ u\| - C\|u\|_{C^4}^2 \\ \partial_\tau \|P_j^- u\| &\le (1 - \frac{j^2}{2})\|P_j^- u\| + C\|u\|_{C^4}^2.\end{aligned} \tag{2.10}$$

By Lemma 2.3, $\|u\|_{C^4}^2 = O(e^{2\lambda\tau})$. By Lemma 2.5, $\|P_j^+ u\| = O(e^{2\lambda\tau})$. Next, if $(1 - j^2/2) > 2\lambda$, then $\|P_j^- u\| = O(e^{(1-j^2/2)\tau})$ by Lemma 2.6. In this case $\|u\|_{L^2} = O(e^{(1-j^2/2)\tau})$, which in turn improves to the decay of $C^{k,\alpha}$ norm (with the same rate) and shows the first part of assertion. Suppose we are in remaining case $(1 - j^2/2) \le 2\lambda$. We have $\|P_j^- u\| = O(e^{2\lambda\tau})$ if $(1 - j^2/2) < 2\lambda$ and $O(\tau e^{2\lambda\tau})$ if $(1 - j^2/2) = 2\lambda$. Then we may apply the same argument with the improved rate $\|u\|_{L^2} = O(e^{2\lambda\tau})$ (or $O(e^{(2\lambda+\epsilon)\tau})$ when $1 - j^2/2 = 2\lambda$). By repeating this finite number of times, we reach at $\|u\|_{C^{k,\alpha}} = O(e^{(1-j^2/2)\tau})$.

FILL THE SECOND PART.

□

Proof of Theorem 1.6. By Theorem 2.4, the condition for Proposition 2.7 is satisfied for $j = 2$. If both c and s are not both zero, then the assertion is satisfied. Otherwise, the condition for Proposition 2.7 is satisfied for $j = 3$. We can iterate this until we find non-zero coefficient. Otherwise the second alternative in Theorem 1.6 holds. □

Remark 2.8. Suppose $\|u\|_{L^2} = O(e^{(1-k^2/2)\tau})$ with $k \geq 2$. By regularity improvement, $\|N(u)\|_{L^2} = O(e^{2(1-k^2/2)\tau})$. Let j be the largest number such that $(1 - j^2/2) > 2(1 - k^2)/2$. Then the same proof argument actually proves slightly stronger asymptotics

$$\|u - \sum_{m=k}^{j}(c_m \cos m\theta + \sin m\theta)e^{(1-m^2/2)\tau}\|_{L^2} = O(\tau e^{2(1-k^2/2)\tau}).$$

Additional τ in the error is due to the possibility that $2(1 - k^2/2) = 1 - m^2/2$ for some m.

3. Unique continuation property

We closely follow the proof in [3].

Proof of Theorem 1.8. We decompose $Nu(\tau) = \sum_k N_{c,k}(\tau)\cos k\theta + N_{s,k}(\tau)\sin k\theta$. Let us denote $\lambda_i = 1 - j^2/2$. We have ODEs

$$\partial_\tau c_k = \lambda_k c_k + N_{c,k}$$

and similar ones for s_k and they yield

$$c_k(\tau) = e^{\lambda_k(\tau-\tau_0)}c_k(\tau_0) + e^{\lambda_k\tau}\int_{\tau_0}^{\tau} e^{-\lambda_k t}N_{c,k}(t)dt.$$

If we send $\tau_0 \to \infty$, then $e^{\lambda_i(\tau-\tau_0)}c_k(\tau_0) \to 0$ due to exponential convergence with arbitrary rate and we obtain another representation

$$c_k(\tau) = -e^{\lambda_k\tau}\int_{\tau}^{\infty} e^{-\lambda_k t}N_{c,k}(t)dt.$$

Let $m \in \mathbb{N}_0$ be a free parameter. We use the first representation if $k \geq m$ and the second representation if $k < m$ and express $e^{-\lambda_m(\tau-\tau_0)}u(\tau)$ as

(3.1)
$$\begin{aligned} e^{-\lambda_m(\tau-\tau_0)}u(\tau) = &\sum_{k\geq m}\left[e^{(\lambda_k-\lambda_m)(\tau-\tau_0)}c_k(\tau_0) + \int_{\tau_0}^{\tau} e^{(\lambda_k-\lambda_m)(\tau-t)}e^{\lambda_m(\tau_0-t)}N_{c,k}(t)dt\right]\cos k\theta \\ &- \sum_{k<m}\left[\int_{\tau}^{\infty} e^{(\lambda_k-\lambda_m)(\tau-t)}e^{\lambda_m(\tau_0-t)}N_{c,k}(t)dt\right] \\ &+ \text{ same expressions for } s_k. \end{aligned}$$

Since $\lambda_k \cos k\theta = L\cos k\theta$, it would be wise to write above as

(3.2)
$$e^{-\lambda_m(\tau-\tau_0)}u(\tau) = e^{(L-\lambda_m)(\tau-\tau_0)}\Pi u(\tau_0) + \int_{\tau_0}^{\tau} e^{(L-\lambda_m)(\tau-t)}e^{\lambda_m(\tau_0-t)}\Pi N u(t)dt - \int_{\tau}^{\infty} e^{(L-\lambda_m)(\tau-t)}e^{\lambda_m(\tau_0-t)}(1-\Pi)Nu(t)dt$$

Here Π is the projection on the eigenspace of eigenvalue less than or equal to λ_m.

For $\tau_0 \leq \tau$, using $L-\lambda_m$ is non-positive definite on the range of Π and $L-\lambda_m$ is positive definite on the range of $I-\Pi$,

(3.3)
$$\left\| \int_{\tau_0}^{\tau} e^{(L-\lambda_m)(\tau-t)}e^{\lambda_m(\tau_0-t)}\Pi N u(t)dt - \int_{\tau}^{\infty} e^{(L-\lambda_m)(\tau-t)}e^{\lambda_m(\tau_0-t)}(1-\Pi)Nu(t)dt \right\|_{L^2}$$
$$\leq \int_{\tau_0}^{\tau} e^{-\lambda_m(t-\tau_0)}\|Nu(t)\|_{L^2}dt + \int_{r}^{\infty} e^{-\lambda_m(t-\tau_0)}\|Nu(t)\|_{L^2}dt$$
$$= \int_{\tau_0}^{\infty} e^{-\lambda_m(t-\tau_0)}\|Nu(t)\|_{L^2}dt.$$

Therefore, for $\tau_0 \leq \tau$

$$e^{-\lambda_m(\tau-\tau_0)}\|u(\tau)\|_{L^2} \leq \|\Pi u(\tau_0)\|_{L^2} + \int_{\tau_0}^{\infty} e^{-\lambda_m(t-\tau_0)}\|Nu(t)\|_{L^2}dt$$
$$\leq \|\Pi u(\tau_0)\|_{L^2} + C\int_{\tau_0}^{\infty} e^{-\lambda_m(t-\tau_0)}\|u(t)\|_{L^2}\|u(t)\|_{H^4}dt$$

For given τ_0,

$$\sup_{\tau\geq\tau_0} e^{-\lambda_m(\tau-\tau_0)}\|u(\tau)\|_{L^2} \leq \|\Pi u(\tau_0)\|_{L^2} + \sup_{\tau\geq\tau_0} e^{-\lambda_m(\tau-\tau_0)}\|u(\tau)\|_{L^2}\left(C\int_{t_0}^{\infty}\|u(t)\|_{H^4}dt\right).$$

We let us choose τ_0 as the first time $C\int_{\tau_0}^{\infty}\|u(t)\|_{H^4}dt \leq 1/2$. Then

$$\sup_{\tau\geq\tau_0} e^{-\lambda_m(\tau-\tau_0)}\|u(\tau)\|_{L^2} \leq 2\|\Pi u(\tau_0)\|_{L^2}.$$

If we take $m \to \infty$, then the right hand side deceases to 0 while $e^{-\lambda_m(\tau-\tau_0)}$ is non-decreasing in m. Therefore, we conclude $\|u(\tau)\|_{L^2} = 0$ for $\tau \geq \tau_0$. i.e. $u = 0$ for $\tau \geq \tau_0$. Indeed, t_0 has to be the initial time smooth solution is defined. Otherwise $\int_{t_0}^{\infty}\|u\|_{C^4}dt = 0$ and we can find smaller $t' < t_0$ such that $\int_{t_0}^{\infty}\|u\|_{C^4}dt < 1/2$, a contradiction.

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References

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