Suppose that a one-parameter family of smooth immersions $\gamma : \mathbb{R}^1 \times [0,T) \to \mathbb{R}^{n+1}$ satisfies

$$\tfrac{\partial}{\partial t}\gamma(z,t) = \tfrac{\partial^2}{\partial s^2}\gamma(z,t) \tag{0.1}$$

eq:CSF

for $z \in \mathbb{R}^1$ and $t \in [0,T)$, where $s$ is the arc length parameter. Then, we call the evolution of a complete immersed curve $\Gamma_t := \gamma(\mathbb{R}^1, t)$ a curve shortening flow.

Notice that if $\gamma$ is a periodic function in $\mathbb{R}$, namely $\gamma(z+L,t) = \gamma(z,t)$, then each image $\Gamma_t$ is a closed curve. Conversely, if the initial image $\Gamma_0$ is a closed curve, then we can find a $L$-periodic immersion $\gamma(\cdot,0)$. Then, by the uniqueness of solution to parabolic PDEs, we can show that $\gamma(0,t)$ remains as a $L$-periodic function, but we won't prove it in this note. Hence, when we study closed curve shortening flow, we consider $\gamma : \mathbb{S}^1 \times [0,T) \to \mathbb{R}^{n+1}$.

In addition, in this winter school, we concentrate on the flows embedded in $\mathbb{R}^2$, namely each $\gamma(\cdot,t)$ is embedding and $\mathbb{R}^{n+1} = \mathbb{R}^2$. If one gets used to the maximum principle through this winter school, then it is very easy to show that a closed curve embedded in $\mathbb{R}^2$ remains embedded under the curve shortening flow.

Now, we present (0.1) as a standard PDE. For each $t \in [0,T)$, the arc length parameter $s$ is defined to satisfy

$$\tfrac{ds}{dz} = \left|\tfrac{\partial}{\partial z}\gamma(z,t)\right|, \tag{0.2}$$

and therefore

$$\tfrac{\partial}{\partial t}\gamma = \tfrac{\partial^2}{\partial s^2}\gamma = \left(\tfrac{dz}{ds}\right)^2 \tfrac{\partial^2}{\partial z^2}\gamma + \tfrac{d^2 z}{ds^2}\tfrac{\partial}{\partial z}\gamma = \left|\tfrac{\partial \gamma}{\partial z}\right|^{-2} \tfrac{\partial^2}{\partial z^2}\gamma - \langle \tfrac{\partial \gamma}{\partial z}, \tfrac{\partial^2 \gamma}{\partial z^2}\rangle \left|\tfrac{\partial \gamma}{\partial z}\right|^{-4} \tfrac{\partial}{\partial z}\gamma. \tag{0.3}$$

eq:standa

Namely, $\gamma(z,t)$ is a vector-valued solution to parabolic type PDEs. Thus, we can employ powerful tools in the theory of differential equations.

**Theorem 0.1** (weak maximum principle)**.** *Suppose that a smooth function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfies $u(x+L,t) = u(x,t) = 0$ for some $L > 0$ and*

$$u_t \leq a u_{xx} + b u_x \tag{0.4}$$

*in $\mathbb{R} \times [0,T)$, where $a, b$ are smooth functions such that $a \geq 0$. Then,*

$$u(x,t) \leq \max_{x \in \mathbb{R}} u(x,0) \tag{0.5}$$

*holds for all $(x,t) \in \mathbb{R} \times [0,T)$.*

*Proof.* For each $\varepsilon > 0$, we define $u^\varepsilon = u - \varepsilon t$ and observe

$$u^\varepsilon_t < u_t \leq a u_{xx} + b u_x = a u^\varepsilon_{xx} + b u^\varepsilon_x. \tag{0.6}$$

eq:eps_pt

We claim that

$$u^\varepsilon(x,t) \leq M := \max_{x \in \mathbb{R}} u(x,0) = \max_{x \in \mathbb{R}} u^\varepsilon(x,0) \tag{0.7}$$

holds in $\mathbb{R} \times [0,T)$. If the claim is true, then passing $\varepsilon$ to zero completes the proof.

Now, towards a contradiction, we suppose that $\sup_Q u^\varepsilon > M$, where $Q := \mathbb{R} \times [0,T)$. Then, given $\delta \in (0, M - \sup_Q u^\varepsilon)$ there exists some space-time point $(x_0,t_0)$ such that $u^\varepsilon < M + \delta$

holds for $\mathbb{R}\times[0,t_0)$ and $u^\varepsilon(x_0,t_0)=M+\delta$. Since $u^\varepsilon(\cdot,t_0)$ attains its maximum at $x_0$, we have $u^\varepsilon_{xx}(x_0,t_0)\leq 0$ and $u^\varepsilon_x(x_0,t_0)=0$. Therefore, combining (0.6) and $a\geq 0$ yields

$$u^\varepsilon_t(x_0,t_0)<0. \tag{0.8}$$

On the other hand, $t\in(0,t_0)$ satisfies $u^\varepsilon_t(x_0,t_0)-u^\varepsilon_t(x_0,t)>0$, which contradicts

$$0>u^\varepsilon_t(x_0,t_0)=\lim_{t\to t_0}\frac{u^\varepsilon_t(x_0,t_0)-u^\varepsilon_t(x_0,t)}{t_0-t}\geq 0. \tag{0.9}$$

□

By using the maximum principle, we can show an exercise problem.

**Exercise 1.** *Let $\Gamma_t$ be a smooth closed curve shortening flow. Then, it develops a singularity at a finite time $T\leq\frac{1}{2}\max_{x\in\Gamma_0}|x|^2$.*

*Proof.* We recall $\gamma:\mathbb{S}^1\times[0,T)\to\mathbb{R}^{n+1}$ satisfying (0.1) and $\Gamma_t=\gamma(\mathbb{S}^1,t)$. Then,

$$\tfrac{\partial}{\partial t}|\gamma|^2=2\langle\gamma,\gamma_t\rangle=2\langle\gamma,\gamma_{ss}\rangle=\tfrac{\partial}{\partial s^2}|\gamma|^2-2|\gamma_s|^2. \tag{0.10}$$

Since $|\gamma_s|=1$ by definition of $s$, the smooth function $u(z,t):=|\gamma(z,t)|^2+2t$ satisfies

$$u_t=u_{ss}=|\gamma_z|^{-2}u_{zz}-\langle\gamma_z,\gamma_{zz}\rangle|\gamma_z|^{-3}u_z \tag{0.11}$$

as in (0.3). In addition, $u(z,t)=u(z+L,t)$ for some $L>0$. Hence, by the maximum principle, we have

$$|\gamma|^2+2t=u\leq\max u(\cdot,0)=\max|\gamma(\cdot,0)|^2=\max_{x\in\Gamma_0}|x|^2=:r^2. \tag{0.12}$$

Therefore, we obtain $|\gamma(\cdot,t)|\leq r^2-2t$ for $t\geq 0$, namely $\Gamma_t$ is contained in the ball of radius $\sqrt{r^2-2t}$. Hence, $\Gamma_t$ is squeezed in the shrinking ball, which disappears at the time $\frac{1}{2}r^2$, namely the singular time satisfies $T\leq\frac{1}{2}r^2$. □

In the most cases, we only use the weak maximum principle, but we provide the strong maximum principle without proofs for the people who are interested in.

**Theorem 0.2** (strong maximum principle)**.** *Suppose that a smooth function $u:\mathbb{R}\times[0,T)\to\mathbb{R}$ satisfies $u(x+L,t)=u(x,t)=0$ for some $L>0$ and*

$$u_t\leq au_{xx}+bu_x \tag{0.13}$$

*in $\mathbb{R}\times[0,T)$, where $a,b$ are smooth functions such that $a>0$. Then,*

$$u(x,t)<\max_{x\in\mathbb{R}}u(x,0) \tag{0.14}$$

*holds for all $(x,t)\in\mathbb{R}\times(0,T)$, unless $u$ is a constant function.*

*Proof.* See *Partial Differential Equations* by Evans. □

Also, we give another variations of the weak maximum principle. The readers are encourages to prove them as exercises.

**Exercise 2** (comparison principle)**.** *Suppose that $L$-periodic smooth functions $u, v : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfy*

$$u_t \le a u_{xx} + b u_x, \qquad\qquad v_t \ge a v_{xx} + v_x, \tag{0.15}$$

*in $\mathbb{R} \times [0,T)$, where $a, b$ are smooth functions such that $a \ge 0$. Moreover $u(x,0) \le v(x,0)$ holds in $\mathbb{R}$. Then,*

$$u(x,t) \le v(x,t) \tag{0.16}$$

*holds for all $(x,t) \in \mathbb{R} \times [0,T)$.*

**Exercise 3** (compactly supported subsolution)**.** *Suppose that a smooth function $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ satisfies*

$$u_t \le a u_{xx} + b u_x \tag{0.17}$$

*in $\mathbb{R} \times [0,T)$, where $a, b$ are smooth functions such that $a \ge 0$. Moreover, $\{x : u(x,t) \neq 0\}$ is a bounded subset in $\mathbb{R}$ for every $t \in [0,T)$. Then,*

$$u(x,t) \le \max_{x \in \mathbb{R}} u(x,0) \tag{0.18}$$

*holds for all $(x,t) \in \mathbb{R} \times [0,T)$.*

We also provide some basic estimates for parabolic PDEs. We recall the Holder norm $C^\alpha$ for $\alpha \in (0,1)$ that

$$\|u\|_{C^\alpha(Q)} = \|u\|_{C^0(Q)} + \sup \left\{ \tfrac{|u(X)-u(Y)|}{d(X,Y)^\alpha} : X, Y \in Q, d(X,Y) \le 1 \right\} \tag{0.19}$$

where $d$ is the parabolic distance given by $d(X,Y)^2 = |x-y|^2 + |t-s|$ for $X = (x,t), Y = (y,s)$. In addition, for $k \in \mathbb{N}$ and $\alpha \in (0,1)$ we define

$$\|u\|_{C^{k,\alpha}(Q)} = \sum_{l=0}^{k} \|u\|_{C^l(Q)} + \sup \left\{ \tfrac{|D^k u(X) - D^k u(Y)|}{d(X,Y)^\alpha} : X, Y \in Q, d(X,Y) \le 1 \right\}. \tag{0.20}$$

**Theorem 0.3** (interior $L^\infty$ estimates)**.** *Suppose that $u : \mathbb{R} \times [0,T) \to \mathbb{R}$ is a smooth $L$-periodic solution to*

$$u_t = a u_{xx} + b u_x + cu \tag{0.21}$$

*where $a, b, c$ are smooth $L$-periodic functions satisfying*

$$\Lambda^{-1} \le a \le \Lambda, \qquad\qquad |b|, |c| \le \Lambda, \tag{0.22}$$

*for some $\Lambda > 0$. Then, for each $\varepsilon > 0$ there exists some constant $C$ only depending on $L, \Lambda, \varepsilon$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R})} \le C \sup_{s \in [t-\varepsilon, t]} \|u(\cdot,s)\|_{L^2([0,L])} \tag{0.23}$$

*holds for $t \in [\varepsilon, T)$.*

*Proof.* See *Second Order Parabolic Differential Equations* by Lieberman. □

**Theorem 0.4** (interior Schauder estimates). *Suppose that $u : Q \to \mathbb{R}$, where $Q = \mathbb{R} \times [0,T)$, is a smooth $L$-periodic solution to*

$$u_t = au_{xx} + bu_x + cu \tag{0.24}$$

*where $a, b, c$ are smooth $L$-periodic functions satisfying*

$$\Lambda^{-1} \leq a, \qquad \|a\|_{C^\alpha(Q)}, \|b\|_{C^\alpha(Q)}, \|c\|_{C^\alpha(Q)} \leq \Lambda, \tag{0.25}$$

*for some $\Lambda > 0$. Then, for each $\alpha \in (0,1)$ and $\varepsilon > 0$ there exists some constant $C$ only depending on $L, \Lambda, \varepsilon, \alpha$ such that*

$$\|u(\cdot,t)\|_{C^{2,\alpha}(Q)} \leq C \sup_{s\in[t-\varepsilon,t]} \|u(\cdot,s)\|_{L^\infty([0,L])} \tag{0.26}$$

*holds for $t \in [\varepsilon, T)$.*

*Proof.* See *Second Order Parabolic Differential Equations* by Lieberman. □

We also recall the Fourier series.

**Theorem 0.5** (Fourier series). *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a $2\pi$-periodic function such that $f \in L^2[-\pi,\pi]$. Then,*

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{0.27}$$

*where*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx. \tag{0.28}$$

*Moreover, $\|f\|^2_{L^2[-\pi,\pi]} = \frac{\pi}{2}a_0^2 + \pi\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$.*

Furthermore, we recall some basic properties of smooth curves in $\mathbb{R}^2$. Let $\gamma$ be a smooth immersion of the smooth curve $\Gamma \subset \mathbb{R}$. Then, the Frenet–Serret formulas yields

$$\mathbf{t} = \tfrac{d}{ds}\gamma, \qquad \tfrac{d}{ds}\mathbf{t} = \kappa\mathbf{n}, \qquad \tfrac{d}{ds}\mathbf{n} = -\kappa\mathbf{t}, \qquad |\mathbf{n}| = |\mathbf{t}| = 1, \tag{0.29}$$

where $s$ is the arc length parameter, $\mathbf{t}$ is the tangent vector, $\mathbf{n}$ is a unit normal vector, $\kappa$ is the curvature. Indeed, one can easily obtain the formulas by differentiating $|\mathbf{n}|^2 = 1$, $|\mathbf{t}|^2 = 1$, $\langle \mathbf{n}, \mathbf{t}\rangle = 0$.

If a smooth embedded curve $\Gamma \subset \mathbb{R}^2$ is a boundary of a convex set $\Omega \subset \mathbb{R}^2$, then the curvature is always positive $\kappa > 0$ and therefore $\mathbf{n} := \kappa^{-1}\frac{d}{ds}\mathbf{t}$ is the unit inward pointing normal vector. Moreover, given $\theta \in \mathbb{R}$ there exists at most one point $x \in \Gamma$ such that $\mathbf{n} = (\cos\theta, \sin\theta)$. Notice that if $\Gamma$ is closed then there exists exactly one such point Therefore, we can reparameterize the angle $\theta$, namely we can let $\gamma(\theta)$ denote the point in $\Gamma$ such that $\mathbf{n} = (\cos\theta, \sin\theta)$. Then, we define a support function[1] $S(\theta)$ by

$$S(\theta) = \langle \gamma(\theta), (\cos\theta, \sin\theta)\rangle. \tag{0.30}$$

[1]One can see the geometric meaning of the support function in the first lecture of prof. Kang.

Thus, we can directly calculate $S_\theta, S_{\theta\theta}$ so that we have

$$\gamma(\theta) = S(\cos\theta, \sin\theta) + S_\theta(-\sin\theta, \cos\theta), \tag{0.31}$$

and

$$\kappa = \frac{1}{S_{\theta\theta} + S}. \tag{0.32}$$

Hence, the support function of a convex curve must satisfy $S_{\theta\theta} + S > 0$. One may consider the following exercises for fun, although it is not required to know the proofs.

**Exercise 4.** *Given a convex closed curve $\Gamma \subset \mathbb{R}^2$, $\inf S > 0$ holds if and only if $\Gamma$ encloses the origin.*

**Exercise 5.** *Suppose that $u : \mathbb{R} \to \mathbb{R}$ is a $2\pi$-periodic smooth function satisfying $u_{\theta\theta} + u > 0$. Then, there exist a closed convex curve $\Gamma \subset \mathbb{R}^2$ whose support function is $u$.*

*Hint.* We define half planes $\mathcal{H}_\theta = \{x \in \mathbb{R}^2 : \langle x, (\cos\theta, \sin\theta)\rangle \leq u(\theta)\}$ and then define a set

$$\Omega = \cap_{\theta\in\mathbb{R}} \mathcal{H}_\theta. \tag{0.33}$$

Then, we can prove that $\Omega$ is a bounded convex set with the smooth boundary $\Gamma = \partial\Omega$, and its support function is $u$. □

Finally, one would be curious if the flow is invariant under parametrizations. We provide a well-known theorem for the independence of parametrizations.

**Theorem 0.6** (reparametrization)**.** *Suppose that a smooth family of immersions $\hat{\gamma} : \mathbb{R}^1 \times [0, T) \to \mathbb{R}^2$ satisfies*

$$\tfrac{\partial}{\partial t}\hat{\gamma}(z, t) = \tfrac{\partial^2}{\partial s^2}\hat{\gamma}(z, t) + V(z, t)\hat{\mathbf{t}}(z, t), \tag{0.34}$$

*where $\hat{\mathbf{t}}(z, t) = |\hat{\gamma}_z|^{-1}\hat{\gamma}_z$ and $V$ is a smooth function. Then, there is a smooth family of diffeomorphisms $\varphi : \mathbb{R}^1 \times [0, T) \to \mathbb{R}^1$ such that $\gamma(\cdot, t) = \hat{\gamma}(\cdot, t) \circ \varphi(\cdot, t)$ satisfies 0.1. In particular, each $\gamma(\cdot, t)$ is an immersions.*

*Proof.* Proposition 1.3.4 in *Lecture Notes on Mean Curvature Flow* by Mantegazza. □