Exercises

Donggun Lee

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A moduli functor $F : Sch^{op} \to Sets$ is called a *fine moduli* if $F \simeq h_X$ for some $X \in Sch$. In this case, we say F is represented by X.

If F is a fine moduli represented by X, then there is a unique family $U \in F(X)$ over X which corresponds to the identity in $h_X(X)$. We call $U \to X$ *the universal family* over X. Note that every family $(T \to S) \in F(S)$ over $S \in Sch$ is the pullback of the universal family along a unique morphism $S \xrightarrow{f} X$, i.e. the map induced $f^* : F(X) \longrightarrow F(S)$ by f sends $(U \to X)$ to $(T \to S)$.

Define $\mathcal{M}_{0,n}$ to be a moduli functor $\mathcal{M}_{0,n} : Sch^{op} \to Sets$ with

$$\mathcal{M}_{0,n}(S) = \{\mathbb{P}^1\text{-bundles over } S \text{ with } n \text{ disjoint sections}\}/\simeq$$

for each $S \in Sch$. Furthermore, define $\overline{\mathcal{M}}_{0,n} : Sch^{op} \to Sets$ to be a moduli functor with

$$\overline{\mathcal{M}}_{0,n}(S) = \{\text{families of rational nodal curves over } S$$
$$\text{with } n \text{ sections, fiberwisely stable}\}/\simeq$$

for each $S \in Sch$.

1. Prove the following
   (a) $\mathcal{M}_{0,2}$ is not a fine moduli (hint: consider Hirzebruch surfaces).
   (b) $\mathcal{M}_{0,3}$ is a fine moduli space represented by a point.
   (c) $\mathcal{M}_{0,n} \simeq (k \setminus \{0,1\})^{n-3} \setminus \Delta$, i.e. $\mathcal{M}_{0,n}$ is a fine moduli represented by the latter space.

2. Let n be a nonnegative integer. Let C be the projective line $\mathbb{P}^1$ or a tree of $\mathbb{P}^1$'s with at worst nodal singularities. Let $p_1, \ldots, p_n \in C$ be n distinct smooth points on the C. Denote

$$\mathrm{Aut}(C, p_i) := \{\sigma \in \mathrm{Aut}(C) \ : \ \sigma(p_i) = p_i \text{ for all } i\}.$$

   (a) Compute the isomorphism class of $\mathrm{Aut}(\mathbb{P}^1, p_i)$ for each n.
   (b) Prove that $\mathrm{Aut}(C, p_i)$ is trivial if and only if each irreducible component has at least three special points, i.e. the nodes or p_i's.

3. Recall that $\overline{\mathcal{M}}_{0,n+1} \to \overline{\mathcal{M}}_{0,n}$ is naturally the universal curve with n sections $\sigma_1, \ldots, \sigma_n$, so each fiber $(\pi^{-1}(p), \sigma_i(p))$ of $p \in \overline{\mathcal{M}}_{0,n}$ is precisely the stable n-pointed curve which p parametrizes. Prove the following.

   (a) There are no singular 3-pointed rational stable curves, so $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \mathrm{pt}$. $(\mathbb{P}^1, (0, 1, \infty))$ is the universal curve of $\overline{\mathcal{M}}_{0,3}$. Conclude that $\overline{\mathcal{M}}_{0,4} \simeq \mathbb{P}^1$. We denote this isomorphism φ.

   (b) There are three distinct singular 4-pointed rational stable curves. These correspond to precisely $0, 1, \infty$ under φ. The isomorphism φ extends the isomorphism $\mathcal{M}_{0,4} \simeq k \setminus \{0, 1\}$ in Problem 1-(c).

   (c) More generally, is the image of σ_i a boundary divisor of $\overline{\mathcal{M}}_{0,n+1}$ for each $i = 1, \ldots, n$? Describe the image.

   (d) Explain how φ realizes a subgroup of $PGL_2 = \mathrm{Aut}(\mathbb{P}^1)$ which is isomorphic to S_4. Compute the images in PGL_2 of all the transitions in S_4.

4. Recall that the blowup B of $\mathbb{A}^2$ along the origin is isomorphic to the closure of $\{((x, y), [x : y]) \in (\mathbb{A}^2 \setminus 0) \times \mathbb{P}^1\}$ in $\mathbb{A}^2 \times \mathbb{P}^1$, with a natural projection to $\mathbb{A}^2$. The fiber $0 \times \mathbb{P}^1$ of the origin is called the exceptional divisor.

   (a) Show that $B \to \mathbb{A}^2$ is an isomorphism away from the exceptional divisor.

   (b) For any subvariety $X \subset \mathbb{A}^2$, we define the *strict transfrom* of X in B to be the closure of $X \setminus 0$ in B. Prove that the strict transforms of two distinct lines through the origin are distinct. More generally, prove that the strict transforms of two smooth curves in $\mathbb{A}^2$ which meets transversally at the origin are disjoint over the origin. Therefore two sections of $\mathbb{A}^2 \to \mathbb{A}^1$ which intersects transversally can be separated by blowing up the intersection points.

   In fact, the exceptional divisor $\mathbb{P}^1$ parametrizes the projectivized tangent vectors at the origin in this case. We also remark that this construction is local so we can define a blowup of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, etc. along a point.

5. Consider the first projection $pr_1 : \overline{\mathcal{M}}_{0,4} \times \mathbb{P}^1 \to \overline{\mathcal{M}}_{0,4}$ with sections $0, 1, \infty, \Delta$ where Δ is given by the commutative diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{0,n} \arrow[r] \arrow[d, "\varphi"', "\simeq"] & \overline{\mathcal{M}}_{0,n} \times \mathbb{P}^1 \arrow[d, "\varphi\times\mathrm{id}"', "\simeq"] \\ \mathbb{P}^1 \arrow[r, "\Delta"] & \mathbb{P}^1 \times \mathbb{P}^1. \end{tikzcd}$$

Then pr_1 is a family of 4-pointed curves which is not necessarily stable, however restricts to the universal curve over $\mathcal{M}_{0,4}$. Construct $\overline{\mathcal{M}}_{0,5}$ by blowing up $\overline{\mathcal{M}}_{0,4} \times \mathbb{P}^1$ along 3 points lying over $\overline{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4} = \{0, 1, \infty\}$ (what should these 3 points be?).

(a) Compute the cohomology of $\overline{\mathcal{M}}_{0,5}$ using the blowup formula.

(b) One can identify the boundary divisors explicitly. For example $D_{4,5}$ is the strict transform of Δ. Identify all the other boundary divisors.

(c) Compute their intersection products. When are they zero?

(d) Prove that the boundary divisors generate $H^*(\overline{\mathcal{M}}_{0,5})$ as a k-algebra.

(e) Let $\pi_i : \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$ be the forgetful morphism which forgets the i-th marking and takes stabilization for each $i = 1, \ldots, 5$. We have the following relations in $H^*(\overline{\mathcal{M}}_{0,5})$:

$$\pi_i^*[\{a\}] = \pi_i^*[\{b\}]$$

for each i and $a, b \in \{0, 1, \infty\} = \overline{\mathcal{M}}_{0,4} \setminus \mathcal{M}_{0,4}$. Show that these relations and the products studied in (b) generate all the relations among boundary divisors.

(f) Compute the S_5-representation of the second (rational) cohomology group of $\overline{\mathcal{M}}_{0,5}$.

(g) Can you find an S_5-invariant basis?

6. (a) Show that a blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ along a point is isomorphic to a blowup $\mathbb{P}^2$ along two distinct points.

(b) Use (a) to conclude that $\overline{\mathcal{M}}_{0,5}$ is isomorphic to a blowup of $\mathbb{P}^2$ along 4 distinct points.

(c) Prove that the 4 points are in general position, i.e. any three of them are not on a line.