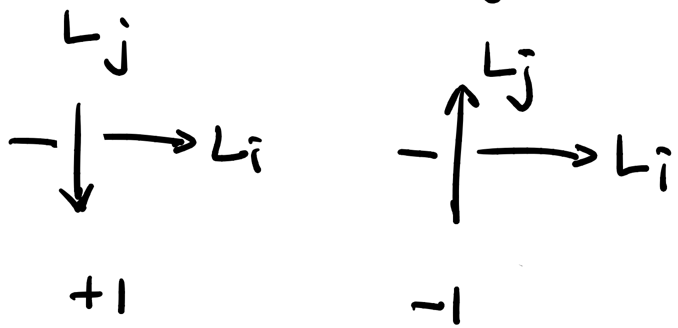
Lecture 2. Kirby Moves

Homology of Y

$\mathcal{L} := (L_1 \sqcup \cdots \sqcup L_n ; p_1, \cdots, p_n)$, an oriented framed link

$lk(L_i, L_j) :=$ the sum of ± 1 over all crossings where L_i crosses under L_j.



OR $\quad H_1(S^3 \setminus L_j) \cong \mathbb{Z}\langle [m] \rangle$

$\Rightarrow [L_i] = lk(L_i, L_j)\,[m]$

Def The <u>linking</u> <u>matrix</u> of $\mathcal{L}$

$= (a_{ij})_{i,j=1,\cdots,n}$

$$a_{ij} = \begin{cases} p_i, & i=j \\ lk(L_i, L_j), & i \neq j \end{cases}$$

E.g.

$\rightsquigarrow \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

Thm $H_1(S^3(\mathcal{L}); \mathbb{Z})$

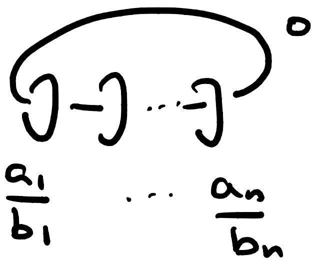
$\cong \operatorname{coker}\left(\text{the linking matrix} \atop \mathbb{Z}^n \to \mathbb{Z}^n\right)$

Cor $S^3(\mathcal{L})$: a homology sphere

$\Longleftrightarrow \det(A) = \pm 1$

Seifert homology sphere

$M = M((a_1, b_1), \cdots, (a_n, b_n))$



By considering $H_1(M)$,

- M is a homology sphere

$\iff a_1 \cdots a_n \sum_{i=1}^{n} b_i / a_i = \pm 1$

E.g. $M((2,-1),(3,1),(5,1))$

$$3 \cdot 5 \cdot (-1) + 2 \cdot 5 \cdot 1 + 2 \cdot 3 \cdot 1 = 1$$

Thm $a_1, \cdots, a_n$: any pairwise relative prime integers $(a_i \geq 2)$

$\Longrightarrow \exists !$ Seifert mfd $M((a_1, b_1), \cdots, (a_n, b_n))$

s.t. $a_1 \cdots a_n \sum_{i=1}^{n} \frac{b_i}{a_i} = 1$

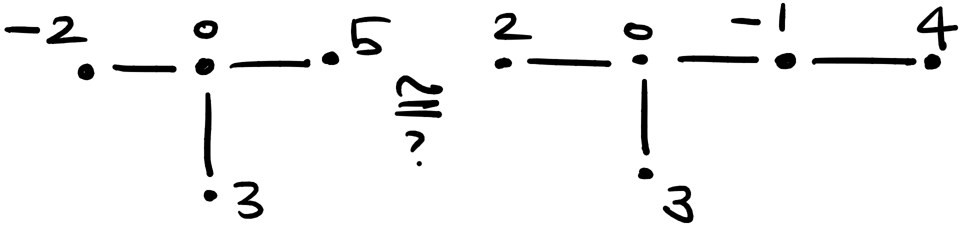
Def $\Sigma(a_1, \cdots, a_n)$ Seifert homology sphere

Rmk $b_1, \cdots, b_n$ is NOT uniquely determined but M's are homeo. each other.

E.g. $\Sigma(2,3,5) = M((2, b_1), (3, b_2), (5, b_3))$

$$15 \cdot b_1 + 10 \cdot b_2 + 6 \cdot b_3 = 1$$

-1	1	1
1	1	-4



Q If $S^3(\mathcal{L}) \cong S^3(\mathcal{L}')$, then how $\mathcal{L}$ & $\mathcal{L}'$ are related?

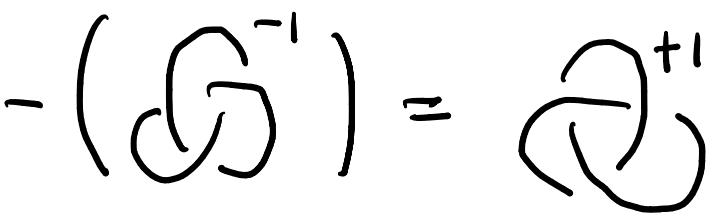
A Kirby moves.

Rmk $Y = S^3(\mathcal{L})$

$\Rightarrow -Y = S^3(\overline{\mathcal{L}})$

$\overline{\mathcal{L}}$: the mirror image of $\mathcal{L}$ & $-$framing of $\mathcal{L}$

E.g.



Thm [Kirby]

$S^3(\mathcal{L}) \cong S^3(\mathcal{L}')$

$\Longleftrightarrow$ $\mathcal{L}'$ can be obtained from $\mathcal{L}$

by a sequence of moves (K1) & (K2)

(K1) Blow up & down

$\mathcal{L} \leftrightsquigarrow \mathcal{L} + \bigcirc^{\pm 1}$

(K2) Handle slides

L_1 & L_2 : two components of $\mathcal{L}$

n_i : the framing of L_i

Slide L_1 over L_2

L_2' : a longitude of $\partial(nb(L_2))$

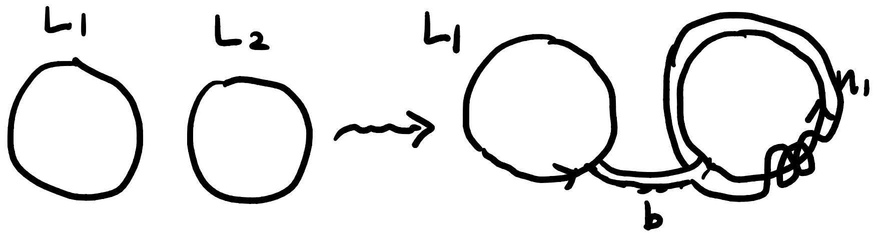
defining the framing n_2 of L_2

$L_1' = L_1 \#_b L_2'$ ← any band connecting L_1 to L_2'

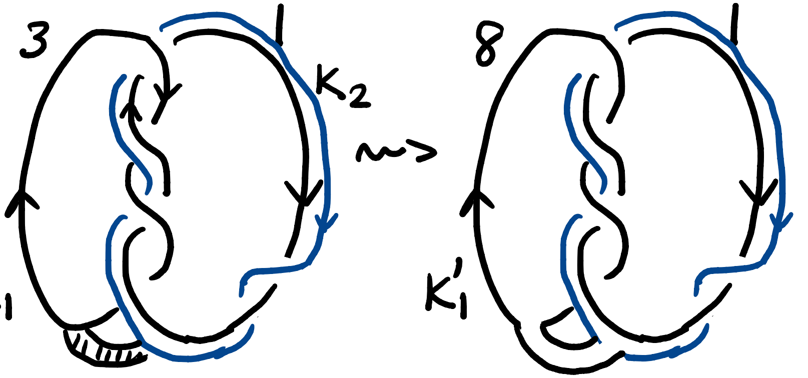
Replace $L_1 \cup L_2$ by $L_1' \cup L_2$

& the new framing of L_1'

$= n_1 + n_2 + 2\,lk(L_1, L_2)$



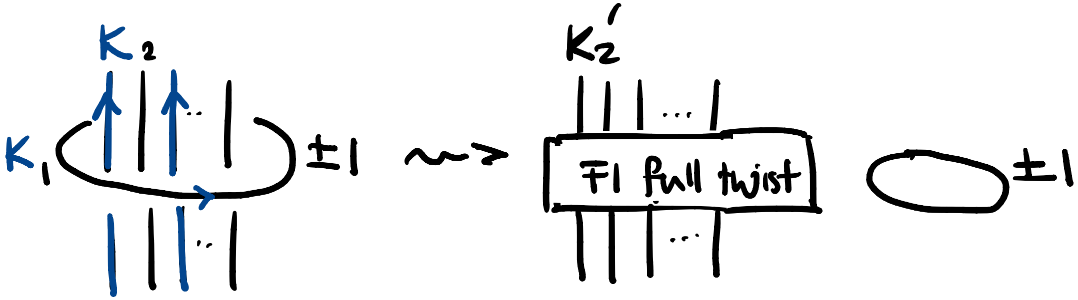
E.g.



$lk(K_1, K_2) = +2$

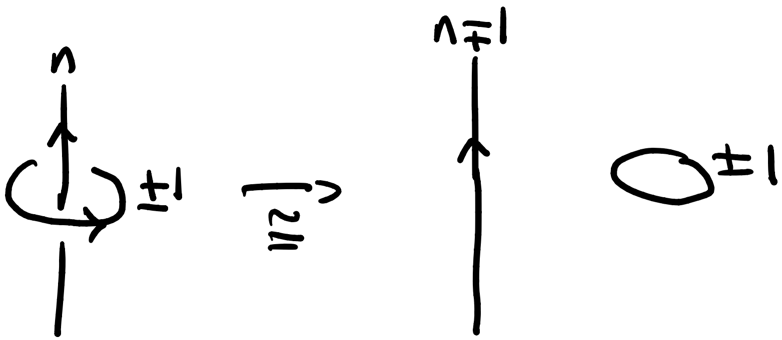
framing of $K_1' = 3 + 1 + 2\cdot 2 = 8$

Prop

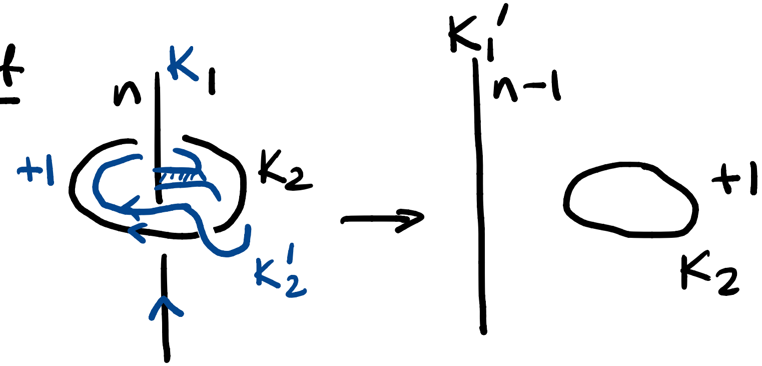


$$fr(K_2') = fr(K_2) \mp lk(K_1, K_2)^2$$

Lemma



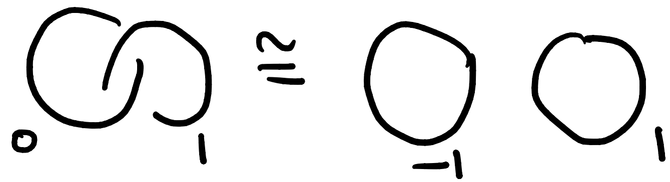
Pf



$$n + 1 + 2\underbrace{lk(K_1, K_2)}_{-1} = n - 1$$

□

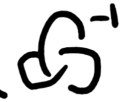
E.g.



$\cong S^3$

E.g. $\Sigma(2,3,5)$

$= M((2,-1),(3,1),(5,1))$

$\cong S^3($  $^{-1})$

$\cong$