Lecture 4. Rohlin invariants & Topology of 4-mfds

W : a s.c. cpt, oriented 4-mfd

with ∂W : a homology sphere

or $\partial W = \phi$

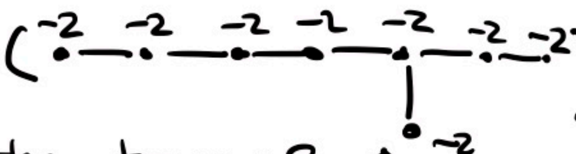
$\rightsquigarrow$ $Q_W : H_2(W) \otimes H_2(W) \longrightarrow \mathbb{Z}$

(Intersection form of W)

is a unimodular, symmetric bilinear form.

Eg. $\cdot$ $X = \pm \mathbb{C}P^2 \rightsquigarrow Q_X = [\pm 1]$

$\cdot$ $X = S^2 \times S^2 \rightsquigarrow Q_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = H$

$\cdot$ $W = W($  $)$

the trace of

$\rightsquigarrow \partial W = \Sigma(2,3,5)$

$Q_W \cong E_8$

$\cdot$ $K3 := \{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \mid z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \}$

K3 surface (Fermat quartic surface)

$\rightsquigarrow Q_{K3} \cong 2E_8 \oplus 3H$

Algebraic aspects of Q

Q : a symmetric bilinear form over $L \cong \mathbb{Z}^n$, $Q : L \times L \to \mathbb{Z}$

- rank$(Q) = rk(Q) = rk_{\mathbb{Z}}(L) = n$

- $L \otimes \mathbb{R} \rightsquigarrow \tilde{Q} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

  $b_+(Q) :=$ # of positive eigenvalues

  $b_-(Q) :=$ # of negative eigenvalues

  ($rk(Q) = b_+(Q) + b_-(Q)$ if Q is nondegenerated)

- Signature of Q

  $$sig(Q) = b_+(Q) - b_-(Q)$$

  If $sig(Q) = b_+(Q)$ or $b_-(Q)$, Q is said to be definite; otherwise, Q is indefinite.

- Type of Q

Def Q is even if

$$Q(v,v) \equiv 0 \pmod 2$$

for $\forall v \in \mathbb{Z}^n$

Otherwise, Q is said to be odd.

Rmk Q is even $\iff Q(e_i, e_i) \equiv 0 \ (2)$

$\{e_1, \cdots, e_n\}$ a basis of L

Alg. classification of unimodular symmetric bilinear form

<table>
<tr><th></th><th>even</th><th>odd</th></tr>
<tr><td>indef.</td><td>$aE_8 \oplus bH$</td><td>$b_+\langle +1\rangle \oplus b_-\langle -1\rangle$</td></tr>
<tr><td>def.</td><td><table><tr><td>rk</td><td>8</td><td>16</td><td>24</td><td>$\cdots$</td></tr><tr><td>#</td><td>E_8</td><td>$\cdots$</td><td>$\cdots$</td><td>$\cdots$</td></tr></table></td><td>$b_+\langle +1\rangle$, $b_-\langle -1\rangle$, $E_8 \oplus \langle -1\rangle$, $\cdots$</td></tr>
</table>

Q : even $\Longrightarrow$ $\mathrm{sig}\, Q \equiv 0 \pmod 8$

Q : indefinite, odd $\Longrightarrow$ $Q \cong b_+\langle +1\rangle \oplus b_-\langle -1\rangle$

Q : " , even $\Longrightarrow$ $Q \cong aE_8 \oplus bH$

Ref. Milnor–Husemoller, "Symmetric bilinear forms"

Question Which symmetric bilinear form is realized by the intersection form of simply-connected, closed topological / smooth 4-mfd?

Thm (Freedman, '82)
Any unimodular form is realized as the intersection form of a simply-connected, closed TOPOLOGICAL 4-mfd.

Thm (Rohlin, '52)
X : a closed simply-connected SMOOTH 4-mfd with even intersection form
$\Longrightarrow$ $\mathrm{sig}\, X \equiv 0 \pmod{16}$

Cor Topological E_8 mfd does NOT admit smooth structure.

Thm (Donaldson, '83)
X : smooth definite
$\Longrightarrow$ $Q_X \cong m\langle +1\rangle$ or $m\langle -1\rangle$

Rohlin invariant

$\lambda : \{\text{homology spheres}\} \to \mathbb{Z}_2$

Y : a homology sphere

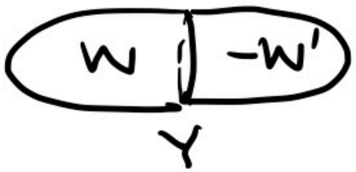
W : an s.c. even 4-mfd with $\partial W = Y$
(such W always exists)

$\rightsquigarrow \mu(Y) := \mathrm{sig}(W)/8 \pmod 2$

Thm λ is well-defined

Pf W' : another even 4-mfd with $\partial W' = Y$

Let $X := W \cup_Y (-W')$
, a closed, oriented 4-mfd.



$$Q_X = Q_{W\#-W'} = Q_W \oplus Q_{-W'}$$

$$\& \ \mathrm{sig}\, X = \mathrm{sig}(W) - \mathrm{sig}(W')$$

By Rohlin's thm,

$$\mathrm{sig}(W) - \mathrm{sig}(W') \equiv 0 \pmod{16}$$

$$\therefore \mathrm{sig}(W) \equiv \mathrm{sig}(W') \pmod{16}$$

$\square$

Properties of μ

- $\mu(S^3) = 0$
- $\mu(-Y) = \mu(Y)$
- $\mu(Y_1 \# Y_2) = \mu(Y_1) + \mu(Y_2)$

E.g. $\mu(\Sigma(2,3,5)) = 1$

$\because \ \mathrm{sig}(E_8) = 8$.

$\Sigma(2,3,5)$

$11/8$ – conjecture

X : simply-connected, closed, oriented

SMOOTH 4-mfd

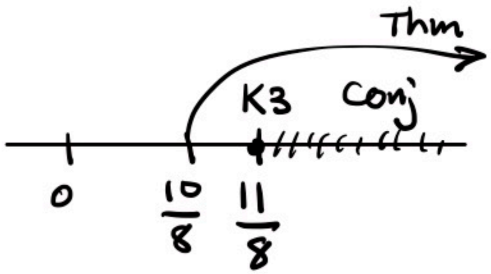
If X : even ($\Longrightarrow Q_X \cong a E_8 \oplus b H$)

<u>Conj</u> ($11/8$ – conj)

$$\frac{\operatorname{rank} Q_X}{|\operatorname{sig} Q_X|} \geq \frac{11}{8}$$

<u>Thm</u> (Furuta, $10/8$ – Thm)

$$\frac{\operatorname{rank} Q_X}{|\operatorname{sig} Q_X|} \geq \frac{10}{8}$$



<u>Note</u>

$Q_{K3} \cong 2E_8 \oplus 3H$

$\Longrightarrow \mathrm{rk} = 16 + 6 = 22$

$\mathrm{sig} = 16$

$\Longrightarrow \dfrac{\mathrm{rk}}{|\mathrm{sig}|} = \dfrac{11}{8}$

Homology cobordism group

Y_1, Y_2 : homology 3-spheres

Def Y_1 & Y_2 are homology cobordant

if $\exists$ a cobordism W from Y_1 to Y_2

s.t. $i_* : H_*(Y_i) \xrightarrow{\cong} H_*(W)$, $i=1,2$.

(i.e. homologically product)

Rmk · Equivalence relation

$$\Theta^3_{\mathbb{Z}} := \left(\{\text{homology spheres}\}/\sim, \# \right)$$

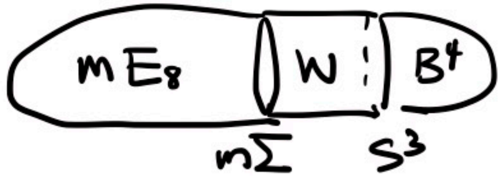
the homology cobordism group

Thm $\lambda : \Theta^3_{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}_2$, epimorphism.

Rmk Until 70's, it was conjectured that this is isom.

Note $\Sigma = \Sigma(2,3,5)$ has infinite order in $\Theta^3_{\mathbb{Z}}$

$$m\Sigma = \underbrace{\Sigma \# \cdots \# \Sigma}_{m} \sim S^3$$



~~$\ast$~~ Donaldson's diagonalization Thm

Triangulation Conjecture

- A closed n-mfd is triangulable $(n \leq 3)$ (i.e $Y \underset{\text{homeo}}{\cong}$ a simplicial complex)
- $\exists$ non triangulable 4-mfd.

Pf Casson inv't.

$\rightsquigarrow$ E_8-mfd is not triangulable $\square$

Triangulation Conjecture

Any closed top. n-mfd $(n \geq 5)$
is triangulable.

Thm [Galewski-Stern, Matumoto]

Conj is true
$\iff \exists$ order 2 element
in $\Theta^3_{\mathbb{Z}}$ s.t. $\mu = 1$

Thm [Malolescu, 2016]

$\nexists$ such element in $\Theta^3_{\mathbb{Z}}$
($\Rightarrow$ Conj is false,
i.e. $\exists$ non-triang. n-mfd
for each $n \geq 5$)

Pf Seiberg-Witten theory
$\square$