담양 연습문제

Day 1

정의:

An <u>algebraic fiber space</u> is a surjective morphism $f: X \to Y$ between smooth projective varieties X & Y with connected fibers $F = f^{-1}(y)$, y = general point of Y.

연습문제 1. $\dim X = \dim Y + \dim F$

연습문제 2. Easy Addition Lemma 의 증명에서

$$\kappa(D_y) + \dim Y \geq \kappa(D)$$

를 다음 경우에 대해서 증명하시오.

① $\kappa(D) = -\infty$

② $\kappa(D) = 0$

연습문제 3. (Iitaka fibration)

Let D be a divisor on a smooth projective variety X such that $\kappa(D) > 0$.

Prove that for all sufficiently large and divisible integer m, $\mathrm{Base}(mD) = \bigcap \{ \mathrm{Supp}\, D' \mid mD \sim D' \geq 0 \}$ is constant.

(It is defined as the stable base locus $SB(D)$.)

Then there is a birational morphism $f: Y \quad X$ such that $f^*(mD) = M + F$ where M is free, F is fixed.

This M defines the Iitaka fibration of D.

Day 2

연습문제 1.

Castelnuovo-Mumford regularity의 정확한 문구를 인터넷을 이용하여 찾으시오.

연습문제 2.

Castelnuovo-Mumford regularity를 이용하여 다음 정리를 증명하시오.

정리 2-3.

X = projective

H = ample Cartier divisor on X such that $|H|$ is free

$\mathcal{F}$ = coherent sheaf on X such that

$$H^i(X, \mathcal{F} \otimes \mathcal{O}_X(-iH)) = 0$$

for any $i > 0$.

Then $\mathcal{F}$ is generated by global sections.

연습문제 3.

Suppose $f: X \to Y$ is a SMOOTH algebraic fiber space. Prove that $f^2: X \times_Y X \longrightarrow Y$ is also a smooth morphism.

Day 3.

연습문제 1.
Popa-Schnell 의 Effective vanishing theorem 을 이용하여 $f^s_* \omega_{X^s/Y}^{\otimes m} \otimes M$ 이 generated by global section ($\forall s \geq 1$) 임을 증명하시오.

$\left(\begin{array}{l} f^s : X \times_Y X \times_Y \cdots \times_Y X \longrightarrow Y \quad \text{s-fold fiber product of } f: X \to Y \\ M = \omega_Y^{\otimes m} \otimes \mathcal{L}^{\otimes m(\dim Y + 1)} \\ \mathcal{L} = \text{ample line bundle such that } M \text{ is free.} \end{array} \right)$

연습문제 2. 강의3에서 쓰인 다음 정리의 증명을 확인.

Lemma 2-5

V = projective variety

$\mathcal{E}$ = locally free sheaf on V, $\mathcal{E} \neq 0$

If there is a line bundle M such that $\mathcal{E}^{\otimes s} \otimes M$ is generated by global sections for any $s > 0$, then $\mathcal{E}$ is nef.

증명:

$\mathcal{E}^{\otimes s} \otimes M$ is globally generated $\forall s > 0$

$\Rightarrow \mathrm{Sym}^s \mathcal{E} \otimes M$ is globally generated $\forall s > 0$

$\Rightarrow \mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(s) \otimes \pi^* M$ is globally generated $\forall s > 0$

$(\pi : \mathbb{P}_V(\mathcal{E}) \longrightarrow V)$

$\Rightarrow \mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(s) \otimes \pi^* M$ is nef

$\Rightarrow \mathcal{O}_{\mathbb{P}_V(\mathcal{E})}(1)$ is nef

$\Rightarrow \mathcal{E}$ is nef (by def)

연습문제 3.

Let U be a smooth quasi-projective variety.
Let X be a smooth projective variety such that U X and D=X-U is a simple normal crossing divisor on X.
Then the Kodaira dimension of U is defined as

$$\kappa(U) = \kappa(\omega_X(D)) = \kappa(K_X + D).$$

Prove that $\kappa(U)$ is independent of the choice of the compactification X.